remark: $e^{i\theta} = \cos\theta + i\sin\theta$ $\quad e^{\cdots} = \sum_{k=1}^{\infty} \frac{(i\theta)^n}{n!} = \quad e^{\cdots}$

Schwarz lemma $\quad f \longleftrightarrow$ rotation

7.1 Def: Automorphism of Ω:

A conformal map from an open set Ω to itself,

$f: \Omega \to \Omega$. $\operatorname{Aut} \Omega$.

group: ① $f, g \in \operatorname{Aut}(\Omega)$, $f \circ g \in \operatorname{Aut} \Omega$

② $(h \circ g) \circ f(z) = h \circ (g \circ f)(z)$

③ $f \circ f^{-1}$, $z \to z$

④ f, f^{-1}

$\mathbb{D} \to \mathbb{D}$ $\qquad \mathbb{D} \to \mathbb{D}$

examples: ① $z \to e^{i\theta} z$ $\qquad$ ② $\psi_\alpha(z) = \dfrac{\alpha - z}{1 - \bar{\alpha} z}$, $|\alpha| < 1$, $\alpha \in \mathbb{C}$

P27 (i) $\psi_\alpha(z)$ is holomorphic ??

$\mathbb{Z}_1$ (ii) $\psi_\alpha(0) = \alpha$, $\psi_\alpha(\alpha) = 0$

(iii) $|z| = 1$, $|\psi_\alpha(z)| = 1$

(iv) f is bijection

Th: $f \in \operatorname{Aut}(\mathbb{D})$, $\exists\, \theta \in \mathbb{R}$, $\alpha \in \mathbb{D}$ s.t. $f(z) = e^{i\theta} \dfrac{\alpha - z}{1 - \bar{\alpha} z}$

proof: $f \in \operatorname{Aut}(\mathbb{D})$, f bijection, f injective, $\exists!\, \alpha \in \mathbb{D}$ s.t. $f(\alpha) = 0$.

$g \triangleq f \circ \psi_\alpha$ $\quad 1^\circ$ $g(0) = 0$ $\quad 2^\circ$ 1° Schwarz lemma

$|g(z)| \le |z|$ $\quad 2^\circ$ $g^{-1} \in \operatorname{Aut}(\mathbb{D})$, $g^{-1}(0) = 0$, $\forall\, w \in \mathbb{D}$, $|g^{-1}(w)| \le |w|$

$w = g(z)$ $\quad |z| \le |g(z)|$ $\qquad 1^\circ, 2^\circ$: $|g(z)| = |z|$ for $\forall\, z \in \mathbb{D}$

⇓

重排 $g(z) = e^{i\theta} z$, Let $z = \psi_\alpha(z)$, ~~$g(z) = g[\psi_\alpha(z)] = f \circ \psi_\alpha \circ \psi_\alpha(z)$~~ :

$\psi_\alpha \circ \psi_\alpha(z) = z \Rightarrow f(z) = e^{i\theta} \psi_\alpha(z)$

Winding . . . let $z=0$, $f(0) = e^{i\theta}\alpha = 0$, $\cancel{e^{i\theta}} \Rightarrow \alpha = 0$

$\alpha \underset{\leftarrow}{\xrightarrow{\varphi_\alpha}} 0 \quad \beta \overset{\varphi_\beta}{\rightleftarrows} 0 \quad \alpha \to \beta: \ \alpha \to 0 \to \beta$

2.2. $\mathbb{D}$ ✓ $\quad \cancel{A \to \mathbb{D} \to \Omega}$

$F \begin{cases} \text{holomorphism} \\ \text{bijection} \end{cases} \Rightarrow \text{isomorphism}$

$\mathbb{H} \xrightarrow{F} \mathbb{D} \xrightarrow{\varphi} \mathbb{D} \xrightarrow{F^{-1}} \mathbb{H}$

$\Gamma(\varphi) = F^{-1} \circ \varphi \circ F$, $\operatorname{Aut}(\mathbb{D}) \xrightarrow{\Gamma} \operatorname{Aut}(\mathbb{H})$

找不到!

$\mathbb{H} \xrightarrow{d} \mathbb{H} \qquad \mathbb{H} \to \mathbb{D} \to \mathbb{D} \to \mathbb{H}$ ✓ ✓ ✓

example: $F(z) = \dfrac{i-z}{i+z}$

2.4: $\operatorname{Aut}\mathbb{H} \leftrightarrow \cancel{M} \ SL_2(\mathbb{R})$

step 1: $\operatorname{Im}\left[f_M(z)\right] = \operatorname{Im}\left(\dfrac{az\bar{z} + bd + adz + bc\bar{z}}{|cz+d|^2}\right) \overset{z = z_1 + iz_2}{=} \dfrac{(ad-bc)\operatorname{Im}(z)}{|cz+d|^2} > 0$

step 2: $f_M \circ f_{M'}(z) = \dfrac{a\frac{ez+f}{gz+h} + b}{c\frac{ez+f}{gz+h} + d} = \dfrac{aez + bgz + af + bh}{(ce+dg)z + cf + dh} = f_{MM'}(z)$

$MM' = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{pmatrix}$

Step 3: $\exists\, M \in \mathcal{G}$, s.t. $\forall z, w \in \mathbb{H}$, $f_M(z) = w$

WTS: $z \in \mathbb{H} \xrightarrow{f} i$, $\exists f$. Setting $d=0$, $\text{Im}(f_{M_1}(z)) = \dfrac{\text{Im}(z)}{|cz|^2}$

Choose $c = \sqrt{\dfrac{z_2}{z_1^2+z_2^2}} > 0$ $\quad \begin{cases} ad-bc=1, \\ d=0 \end{cases}$ $\therefore$ $b = -c^{-1}$, let $a=0$

$\text{Im}(f_M(z)) = 1$

$M_1 = \begin{pmatrix} 0 & -c^{-1} \\ c & 0 \end{pmatrix}$. Let $f_{M_1}(z) = u_1 + i$, $b = -u_1$, build $M_2 = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$

$f_{M_2}[f_{M_1}(z)] = i$, Thus $f_{M_2M_1}(z) = i$

Let F ...

Step 4: $\exists$ $g(z) = e^{-2i\theta} z$, $F \circ f_{M_\theta} \circ F^{-1} = g$ $\quad M = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

Step 5:

Suppose $f \in \text{Aut}(\mathbb{H})$, s.t. $f(\beta) = i$

$N \in \mathcal{G}$, $f_N(i) = \beta$, Let $g = f \circ f_N$, $g(i) = i$

Then $F \circ g \circ F^{-1}(0) = 0$, $\xrightarrow{\text{Schwarz Lemma}}$ $F \circ g \circ F^{-1}$ is rotation.

Step 4: $\exists\, f_{M_\theta}$ s.t. $F \circ T(g) = T(f_{M_\theta})$, $g = f_{M_\theta}$

$f = f_{M_\theta N^{-1}}$