复分析讨论班第八次讨论

ZiJie Wen

Department of Mathematics, Sustech

2023.2.1

Something Removable

Theorem 3.1 (Riemann's theorem on removable singularity)

Suppose that f is holomorphic in an open set Ω except possibly at a point z_0 in Ω. If f is bounded on $\Omega - \{z_0\}$, then z_0 is a removable singularity.

go further

Corollary 3.2

Suppose that f has an isolated singularity at the point z_0. Then z_0 is a pole of f if and only if $|f(z)| \to \infty$ as $z \to z_0$.

Isolated singularities

- Removable singularities (f bounded near z_0)
- Pole singularities ($|f(z)| \to \infty$ bounded near z_0)
- Essential singularities

Not Removable and Not Pole

Theorem 3.3 (Casorati-Weierstrass)

Suppose f is holmorphic in the punctured disc $D_r(z_0) - \{z_0\}$ and has an essential singularity at z_0. Then, the image of $D_r(z_0) - \{z_0\}$ under f is dense in the complex plane.

Definition

Def. meromorphic

A function f on a open setΩ is **meromorphic** if there exist a sequence of point $\{z_0, z_1, z_2, \dots\}$ that has no limit points on Ω, and such that
(i) the function f is holomorphic in $\Omega - \{z_0, z_1, z_2, \dots\}$, and
(ii) f is poles at the points $\{z_0, z_1, z_2, \dots\}$.

Def. pole(essential/removable) at infinity

If $F(z) = f(1/z)$ has a pole(essential/removable) at the origin.

the final of meromorphic is rational

Def. meromorphic in the extended complex

A meromorphic function in the complex plane that is either holomorphic at infinity or has a pole at infinity is said to **meromorphic in the extended complex plane**.

Theorem 3.4

The meromorphic functions in the extended complex plane are the rational functions.

White board

Some definition of Riemann Sphere

Riemann Sphere

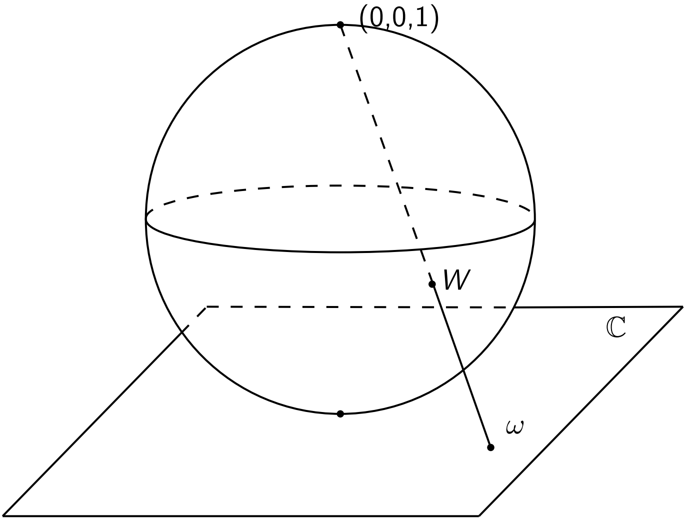
We denote $\mathbb{S}$ in the Euclidean space $\mathbb{R}^3$ with coordinates(X, Y, Z) where the XY-plane is identified with $\mathbb{C}$.
We denote by $\mathbb{S}$ the sphere centered at $(0, 0, \frac{1}{2})$ and of radius $1/2$.
From $\mathbb{S}$ to $\mathbb{R}$,

$$x = \frac{X}{1 - Z} \text{ and } y = \frac{Y}{1 - Z}$$

From $\mathbb{R}$ to $\mathbb{S}$,

$$X = \frac{x}{x^2 + y^2 + 1}, \; Y = \frac{y}{x^2 + y^2 = 1}, \; Z = \frac{x^2 + y^2}{x^2 + y^2 + 1}$$

Riemann Sphere



Thanks for listening !