复分析讨论班第七次讨论

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Ready for start

Quick Review

- holomorphic function
- Coursat's theorem
- Cauchy's theorem
- Cauchy's integral formulas
- Some applications

Start for today's journal

Classification of singularity

- removable singularity
- pole
- essential singularity

Thinking

Remark

By Cauchy's theorem, a holomorphic function f in an open set which contains a closed curve γ and its interior satisfies

$$\int_\gamma f(z)dz = 0$$

but, by Chapter1 exercise 25 we know that

$$\int_C \frac{1}{z}dz = 2\pi i$$

Question

What happens if f has a pole in the interior of the curve? that is what we want to know in this section.

Some definition

Def. Point singularity (Isolated singularity)

It is a complex number such that f is defined in a neighborhood of z_0 but not at the point z_0 itself.

Def. Zero

A complex number z_0 is a **Zero** for the holomorphic function f if $f(z_0) = 0$.

Zeros

Lemma

The zeros of a non-trivial holomorphic function are isolated.

证明.

If f is holomorphic in Ω and $f(z_0) = 0$ for some $z_0 \in \Omega$.
If for every neighborhood U of z_0, there exist a point $p \in U - \{z_0\}$ such that $f(p) = 0$.
This allow us to choose a sequence $\{z_n\}$ by choosing z_n in $\{z \in \Omega | 0 < |z - z_0| < z\}$ such that $f(z_n) = 0$.
By Chapter2 Theorem4.8, we know f is constant. □

Theorem of zero

Theorem 1.1

Suppose that f is holomorphic in a connected open set Ω, has a zero at a point $z_0 \in \Omega$, and does not vanish identically in Ω. Then there exists a neighborhood $U \subset \Omega$ of z_0, a non-vanishing holomorphic function g on U, and a unique positive integer n such that

$$f(z) = (z - z_0)^n g(z) \text{ for all } z \in U.$$

Go further

Remark

In theorem above, the n is unique and only relate to the point z_0. It may allow us to consider n as a property depend on the choose of the point z_0 and the function f.

Def. **order of zero**

In the case of above theorem, we say that f has a **zero of order** n(or **multiplicity** n).

Def. **simple**

If a zero is of order 1, we say that it is **simple**.

Poles

Def. deleted neighborhood of z_0

It is a open disc centered at z_0, minus the point z_0, that is, the set

$$\{z : 0 < |z - z_0| < r\}$$

Def. pole

We say that a function f defined on a deleted neighborhood of z_0 has a **pole** at z_0, if the function $1/f$, defined to be zero at z_0, is holomorphic in a full neighborhood of z_0.

Extend the theorem of zeros to poles

Observation

From the definition, the pole z_0 of a function f, which is holomorphic in the neighborhood of z_0, is a zero of $1/f$. Thus, we can extend last theorem to the case of pole.

Theorem 1.2

If f has a pole at $z_0 \in \Omega$, then in a neighborhood of that point there exist a non-vanishing holomorphic function h and a unique positive integer n such that

$$f(z) = (z - z_0)^{-n} g(z)$$

Extension for the theorem of poles

Def. **order of pole**

In the case of above theorem, we say that f has a **zero of order** n(or **multiplicity** n).

Def. **simple**

If a pole is of order 1, we say that it is **simple**.

Theorem of poles

Review Chapter 2 Theorem 4.4

Suppose f is holomorphic in a open set Ω. If D is a disc centered at z_0 and whose closure is contained in Ω.

$$\forall z \in D, f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n, a_n = \frac{f^{(n)}(z_0)}{n!}$$

Theorem 1.3

If f has a pole of order n at z_0, then

$$f(z) = \frac{a_{-n}}{(z-z_0)^n} + \frac{a_{-n+1}}{(z-z_0)^{n-1}} + \cdots + \frac{a_{-1}}{(z-z_0)} + G(z),$$

where G is a holomorphic function in a neighborhood of z_0.

Residue

Observation

If C is any circle centered at z_0, and only contained a pole z_0, then we get

$$\frac{1}{2\pi i}\int_C f(z)dz = a_{-1}$$

Def. **residue**

We define the **residue** of f at z_0 as the coefficient a_{-1} as above.

Theorem of poles

Theorem 1.4

If f has a pole of order n at z_0, then

$$\mathrm{res}_{z_0} f = \lim_{z \to z_0} \frac{1}{(n-1)!} \left(\frac{d}{dz} \right)^{n-1} (z - z_0)^n f(z).$$

where G is a holomorphic function in a neighborhood of z_0.

Thanks for listening !