复分析讨论班第四次讨论

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Review

Remark

From the last chapter, we learn Cauchy theorem. We knows that if f is holomorphic in a open set Ω, then for every triangle $T \subset \Omega$ whose interior is also contained in Ω, then $\int_T f(z)\mathrm{d}x$.

When triangle vanish

Question

Now, we what to measure the gap from the result to the condition. That is, when the function f is holomorphic?

Morera 's Theorem

Suppose f is a continuous function in the open disc D such that for any triangle T contained in D

$$\int_T f(z)\,dz = 0$$

then f is holomorphic.

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The past is its prelude

Similar Theorem(15.3.1)

For a real-value function sequence $\{f_n\}_{n=1}^{\infty}$, if $\forall n \in \mathbb{Z}_+$, f_n is continuous on I, and $\{f_n\}_{n=1}^{\infty}$ uniformly converge to f, then f is continuous on I.

holomorphic for a convergent

Observation

In real-value function, the uniformly convergent can hold continuity. There is similarity in complex-value function. Actually, "holomorphic" is more rigid than the "derivable". This allow us to go further to the holomorphic.

Theorem

If $\{f_n\}_{n=1}^{\infty}$ is a sequence of holomorphic functions that converges uniformly to a function f in every compact subset of Ω, then f is holomorphic in Ω.

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To compute the derivation

further

Sometimes, the function f cannot be expressed by common symbols. We wish that we can compute the derivative of f through the sequence $\{f_n\}_{n=1}^{\infty}$.

Remark

In mathematical analysis, we have proposition below(15.3.7):
(a) $\forall n \in \mathbb{Z}_+, f_n \in C^1[a, b]$.
(b) $\lim_{n\to\infty} f_n' = g$
(c) $\exists x_0 \in [a, b], s.t.\{f_n(x_0)\}_{n=1}^{\infty}$ converge
then , $\{f_n\}_{n=1}^{\infty}$ uniformly converge to some function f and

$$(\lim_{n\to\infty} f_n)' = \lim_{n\to\infty} f_n'$$

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Observation

According to the work for Chauchy 's theorem, we know the derivation of f on z_0 can be computed by the value of f in a small circle C whose center is z_0.

Theorem

If $\{f_n\}_{n=1}^{\infty}$ is a sequence of holomorphic functions that converges uniformly to a function f in every compact subset of Ω, then the sequence of $\{f_n'\}_{n=1}^{\infty}$ converge uniformly to f' on every compact subset of Ω.

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Some Function constructed by a sequence

$$F(x) = \sum_{n=1}^{\infty} f_n(x)$$

Common examples

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

$$\vartheta(t) = \sum_{n=-\infty}^{+\infty} e^{-\pi n^2 t}$$

1. Morera's theorem
2. Sequence of holomorphic functions
3. Holomorphic functions defined in terms of integrals

Further step to generality

Remark

$$\varphi(x) = \int_a^b F(x, \xi) \, \mathrm{d}\xi$$

The final stop

Theorem

Let $F(z,s)$ be defined for $(z,s) \in \Omega \times [0,1]$ where Ω is an open set in $\mathbb{C}$. Suppose F satisfies the following properties:
(i) $F(z,s)$ is holomorphic in z for each s.
(ii) F is continuous on $\Omega \times [0,1]$ by

$$f(x) = \int_0^1 F(z,s)\,\mathrm{d}s$$

is holomorphic.

Thanks!