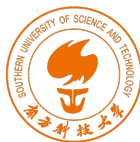


复分析讨论班第四次讨论

Zijie Wen

Department of Mathematics, Sustech

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- ① Morera's theorem
- ② Sequence of holomorphic functions
- ③ Holomorphic functions defined in terms of integrals

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Review

Remark

From the last chapter, we learn Cauchy theorem. We know that if f is holomorphic in a open set Ω , then for every triangle $T \subset \Omega$ whose interior is also contained in Ω , then $\int_T f(z)dx$.

When triangle vanish

Question

Now, we what to measure the gap from the result to the condition.
That is, when the function f is holomorphic?

Morera 's Theorem

Suppose f is a continuous function in the open disc D such that
for any triangle T contained in D

$$\int_T f(z) dz = 0$$

then f is holomorphic.

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The past is its prelude

Similar Theorem(15.3.1)

For a real-value function sequence $\{f_n\}_{n=1}^{\infty}$, if $\forall n \in \mathbb{Z}_+$, f_n is continuous on I , and $\{f_n\}_{n=1}^{\infty}$ uniformly converge to f , then f is continuous on I .

holomorphic for a convergent

Observation

In real-value function, the uniformly convergent can hold continuity. There is similarity in complex-value function. Actually, "holomorphic" is more rigid than the "derivable". This allow us to go further to the holomorphic.

Theorem

If $\{f_n\}_{n=1}^{\infty}$ is a sequence of holomorphic functions that converges uniformly to a function f in every compact subset of Ω , then f is holomorphic in Ω .

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If $\{f_n\}_{n=1}^{\infty}$ is a sequence of holomorphic functions that converges uniformly to a function f in every compact subset of Ω , then f is holomorphic in Ω .

To compute the derivation

further

Sometimes, the function f cannot be expressed by common symbols. We wish that we can compute the derivative of f through the sequence $\{f_n\}_{n=1}^{\infty}$.

Remark

In mathematical analysis, we have proposition below(15.3.7):

(a) $\forall n \in \mathbb{Z}_+, f_n \in C^1[a, b]$.

(b) $\lim_{n \rightarrow \infty} f'_n = g$

(c) $\exists x_0 \in [a, b], s.t. \{f_n(x_0)\}_{n=1}^{\infty}$ converge

then , $\{f_n\}_{n=1}^{\infty}$ uniformly converge to some function f and

$$\left(\lim_{n \rightarrow \infty} f_n\right)' = \lim_{n \rightarrow \infty} f'_n$$

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Observation

According to the work for Cauchy's theorem, we know the derivation of f on z_0 can be computed by the value of f in a small circle C whose center is z_0 .

Theorem

If $\{f_n\}_{n=1}^{\infty}$ is a sequence of holomorphic functions that converges uniformly to a function f in every compact subset of Ω , then the sequence of $\{f'_n\}_{n=1}^{\infty}$ converge uniformly to f' on every compact subset of Ω .

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Some Function constructed by a sequence

$$F(x) = \sum_{n=1}^{\infty} f_n(x)$$

Common examples

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

$$\vartheta(t) = \sum_{n=-\infty}^{+\infty} e^{-\pi n^2 t}$$

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Further step to generality

Remark

$$\varphi(x) = \int_a^b F(x, \xi) d\xi$$

The final stop

Theorem

Let $F(z, s)$ be defined for $(z, s) \in \Omega \times [0, 1]$ where Ω is an open set in \mathbb{C} . Suppose F satisfies the following properties:

- (i) $F(z, s)$ is holomorphic in z for each s .
- (ii) F is continuous on $\Omega \times [0, 1]$ by

$$f(x) = \int_0^1 F(z, s) \, ds$$

is holomorphic.

Thanks!