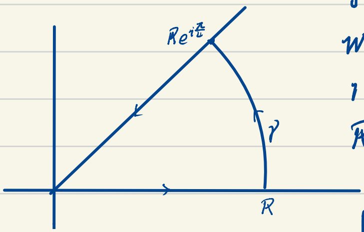
1. $\int_0^\infty \sin x^2\, dx = \int_0^\infty \cos x^2\, dx = \frac{\sqrt{2\pi}}{4}$

[Sol]:

$\int_{-\infty}^{+\infty}\cos x^2\, dx = \int_{-\infty}^{+\infty}\sin x^2\, dx = 2\int_0^{+\infty}\sin x^2\, dx \Rightarrow \int_0^\infty \sin x^2\, dx = \int_0^\infty \cos x^2\, dx$



$\oint e^{-z^2}dz = \int_0^R e^{-x^2}dx + \int_0^{\frac{\pi}{4}} e^{-R^2(\cos 2\theta + i\sin 2\theta)}\, iRe^{i\theta}d\theta + \int_R^0 e^{-x^2 e^{2i\frac{\pi}{4}}}\, e^{i\frac{\pi}{4}}dx = 0.$

where $\left|\int_0^{\frac{\pi}{4}} e^{-R^2(\cos 2\theta + i\sin 2\theta)}\, iRe^{i\theta}d\theta\right| \le \int_0^{\frac{\pi}{4}} e^{-R^2\cos 2\theta}R\, d\theta \to 0$

Hence $\int_0^R \left(\frac{\sqrt2}{2}+i\frac{\sqrt2}{2}\right) e^{-ix^2}dx = \int_0^R e^{-x^2}dx$

$R\to+\infty$: $\int_0^\infty\left(\frac{\sqrt2}{2}+i\frac{\sqrt2}{2}\right)(\cos x^2 - i\sin x^2)\,dx = \frac{\sqrt\pi}{2}$

$Re\left(\int_0^\infty\left(\frac{\sqrt2}{2}+i\frac{\sqrt2}{2}\right)(\cos x^2 - i\sin x^2)\,dx\right) = \frac{\sqrt2}{2}\left(\int_0^\infty \cos x^2+\sin x^2\, dx\right) = \frac{\sqrt\pi}{2}.$

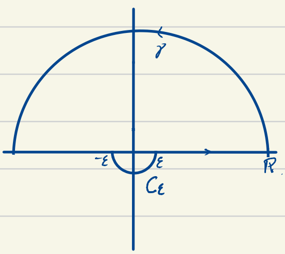
$\int_0^\infty \sin x^2\, dx = \int_0^\infty \cos x^2\, dx = \frac{\sqrt{2\pi}}{4}$ □

idea: $\sin x \to e^{ix}-e^{-ix}$ 对称 e^{ix}

2. $\int_0^\infty \frac{\sin x}{x}dx = \frac{\pi}{2}$.

[Sol]: $\frac{1}{2i}\int_{-\infty}^{+\infty}\frac{e^{ix}-1}{x}dx = \frac{1}{2i}\int_{-\infty}^{+\infty}\frac{\cos x + i\sin x - 1}{x} = \frac{1}{2i}\left(\int_{-\infty}^0 + \int_0^{+\infty}\right) \frac{\cos x}{x} + \frac{i\sin x}{x} - \frac{1}{x}\, dx$

$= \frac{1}{2i}\int_0^{+\infty}\frac{2i\sin x}{x}dx = \int_0^{+\infty}\frac{\sin x}{x}dx.$



$\oint \frac{1}{2i}\frac{e^{iz}-1}{z}dz = \left(\int_{-R}^{-\varepsilon}+\int_\varepsilon^R+\int_{C_\varepsilon}+\int_\gamma\right)\frac{e^{iz}-1}{2iz}dz = 0$ 0 is removable

where $\left(\int_{-R}^{-\varepsilon}+\int_\varepsilon^R\right)\frac{e^{iz}-1}{2iz}dz \to \int_0^{+\infty}\frac{\sin x}{x}dx$ $(R\to\infty)$

$\lim_{z\to0}\frac{e^{iz}-1}{2iz} = \frac{ie^0}{2i} = \frac{1}{2} \Rightarrow \int_{C_\varepsilon}\frac{e^{iz}-1}{2iz}dz\to0$ $(\varepsilon\to0)$

$\left|\int_\gamma \frac{e^{iz}}{2iz}dz\right| = \left|\int_0^\pi \frac{e^{iR\cos\theta}e^{-R\sin\theta}}{2iRe^{i\theta}}\, iRe^{i\theta}d\theta\right| \le \int_0^\pi\frac{e^{-R\sin\theta}}{2}d\theta\to0$

Hence $\int_0^{+\infty}\frac{\sin x}{x}dx = \int_0^\pi \frac{1}{2iz}dz = \frac{\pi}{2}$ □

4. $e^{-\pi\xi^2} = \int_{-\infty}^{+\infty} e^{-\pi x^2}e^{2\pi i x\xi}dx$

[Sol]: $\int_{-\infty}^{+\infty} e^{-\pi(x-\xi)^2}e^{-\pi\xi^2}dx = e^{-\pi\xi^2}\int_{-\infty}^{+\infty}e^{-\pi(x-\xi)^2}dx$

It suffices to show $\int_{-\infty}^{+\infty}e^{-\pi(x+i\xi)^2}dx = 1$. $\forall\xi\in\mathbb{R}$. for $\int_{-\infty}^{+\infty}e^{-\pi(x-r)^2}dx = \int_{-\infty}^{+\infty}e^{-\pi x^2}dx$.

By Ex. 1 in Chapter 2 of Stein's complex analysis, we have this proposition.

5.

[Sol]: suppose $f(z) = u(x,y)+iv(x,y)$.

C-R : $\begin{cases} \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \end{cases}$

By Green's theorem: $\int_T u(x,y)dx - v(x,y)dy = \iint_{\text{Interior of } T} \frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\,dxdy = 0$

$\int_T u(x,y)dy + v(x,y)dx = \iint_{\text{Interior of } T} \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\, dxdy = 0$

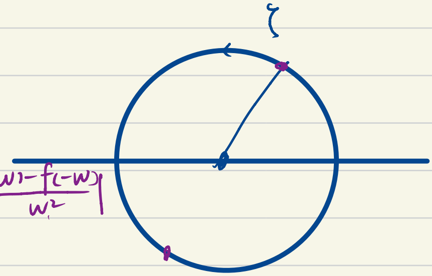
Hence

$\int_T f(z)dz = \int_T u(x,y)dx - v(x,y)dy + i\int_T u(x,y)dy + v(x,y)dx = 0$ □

7.

[Proof]: $f(z) = \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{(\zeta - z)^2} d\zeta$

$f(0) = \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{\zeta^2}d\zeta = \frac{1}{2\pi i}\int_C -\frac{f(\pi)}{\pi^2} d\zeta$. $(\pi = \zeta^{-1})$.



$\Rightarrow |2f(0)| = \left|\frac{1}{2\pi i}\int_C \frac{f(\zeta)-f(-\zeta)}{\zeta^2}d\zeta\right| \le$

$= 2\pi r \cdot \left|\frac{f(w)-f(-w)}{w^2}\right|$

$\le \frac{r}{r^2} d = \frac{d}{r} \sim d$.

$r \to 1$.

$f(z) = a_0 + a_1 z$

$f(w) - f(-w) = F(w)$,

$|F(w)| < d$.

$\left|\frac{f(w)-f(-w)}{w^2}\right| \le \frac{d}{r^2}$

$\left|\frac{f(\cdot)-f(-\cdot)}{\cdot^2}\right|$

① $|w| = r$

$\left|\frac{1}{2\pi i}\int_C \frac{f(\zeta)-f(-\zeta)}{\zeta^2} d\zeta\right| = \left|\frac{f(w)-f(-w)}{w^2} \cdot r\right| = \left|\frac{f(w)-f(-w)}{r}\right| = d$.

$\left|\frac{1}{r}\int_C f(\zeta) - f(-\zeta)\, d\zeta\right|$