In the previous chapters, we discussed some analytical results of holomorphic functions, and now we want to use the mapping properties of holomorphic functions to analyze some more geometric properties.

When encounter the complex plane, we've already defined the topology of C, and a natural consideration is to explore the "homeomorphic relation" between some open sets.

Homeomorphic function $f$: $\underbrace{f \text{ a continuous bijection}}_{①}$ and $\underbrace{f^{-1} \text{ continuous}}_{②}$

but we know ① $\not\Rightarrow$ ② since $f: [0,1) \to S^1$; $f(t) = e^{i2\pi t}$ is a counter-example.

Since we want more analytical properties than topological ones, and we know that holomorphic functions have very strong properties

We want to describe a new relation:
(stronger than "homeomorphic")

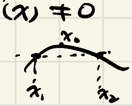
Conformally equivalent (or biholomorphic)

defined by "conformal map" $f$ := a bijective holomorphic function $f: U \to V$, where $U, V$ open in $\mathbb{C}$. then call $U, V$ conformally equivalent.

First: $f$ holomorphic bijection $\Rightarrow$ $f'$ doesn't vanish $\Rightarrow$ $f^{-1}$ holomorphic

**Proposition 1.1** *If $f : U \to V$ is holomorphic and injective, then $f'(z) \neq 0$ for all $z \in U$. In particular, the inverse of $f$ defined on its range is holomorphic, and thus the inverse of a conformal map is also holomorphic.*

联想: In $\mathbb{R}$. if $f$ injective and differentiable in $[a,b]$, then $f'(x) \neq 0$
(otherwise




$f'(x_0) = 0$
$\Rightarrow f(x_1) = f(x_2)$)

the idea is exactly the same.

(local is enough)

proof: Otherwise suppose $z_0 \in \Omega$ s.t. $f'(z_0) = 0$

simple and concrete — abstract

the $f$ holomorphic $\Rightarrow$ near $z_0$, $f(z) - f(z_0) = \underline{a(z-z_0)^k} + \underline{G(z)}$ (with $k \geq 2$)

$G(z) = o((z-z_0)^{k+1})$

we want to gain the different solution of L.H.S. (thus R.H.S)

thus we want use Rouché's theorem

**Theorem 4.3 (Rouché's theorem)** *Suppose that $f$ and $g$ are holomorphic in an open set containing a circle $C$ and its interior. If*

$$|f(z)| > |g(z)| \quad \text{for all } z \in C,$$

*then $f$ and $f+g$ have the same number of zeros inside the circle $C$.*

but $G$ vanishing faster than $a(z-z_0)^k$

$\Rightarrow$ 同减 $w$ (微小常量)

$$f(z) - f(z_0) - w = \underbrace{(a(z-z_0)^k - \underline{w})}_{F(z)} + G(z)$$

thus $\exists$ a circle center at $z_0$ with enough small radius such that

$|G(z)| < |F(z)|$ on the circle $\Rightarrow$ $\underline{F} \xrightarrow{\text{same zeros}} \underline{F+G}$

$F = 0 \Leftrightarrow (z-z_0)^k = \frac{w}{a}$ $(k \geq 2)$

$\Downarrow$

$z_1, z_2$ s.t. $f(z_i) - f(z_0) - w = 0$ $\Leftarrow$ at least two different zeros $z_1, z_2$

$\Downarrow$

$f$ not injective $\Rightarrow$ contradicted!

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[as $f^{-1\prime}(w) = \frac{1}{f'(f^{-1}(w))}$ and $f' \neq 0$ $\Rightarrow$ $f^{-1\prime}$ exists $\Rightarrow$ $f^{-1}$ holomorphic]

Thus it's easily to see that

① $f$ a conformal map $\Rightarrow$ $f$ a homeomorphic function (holomorphic $\Rightarrow$ continuous)

② conformally equivalent is a equivalance relation.

$$\begin{cases} U \longrightarrow U \quad f(z) = z \\ U \xrightarrow{f} V \Rightarrow V \xrightarrow{f^{-1}} U \\ U \xrightarrow{f} V \xrightarrow{g} W \Rightarrow U \xrightarrow{g \circ f} W \end{cases}$$ (the 链式法则?)

③ Conformally equivalent could be expressed more precisely

$\exists f, g$ holomorphic, $f: U \to V$ $g: V \to U$ s.t. $g(f(z)) = z$ and $f(g(\omega)) = \omega$ for $\forall z \in U$ and $\omega \in V$

Remark: the $f'(z) \neq 0$ is the condition to "preserve angles" 保角 (Conformal)

but our convention of conformal

biholomorphic ($\Rightarrow f'(z) \neq 0$)

(stronger)

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**The first Example**: $\mathbb{D} := \{ z \in \mathbb{C} : |z| < 1 \}$

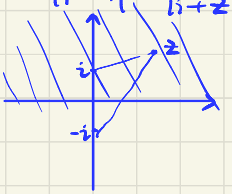
$\mathbb{H} := \{ z \in \mathbb{C} : \mathrm{Im}(z) > 0 \}$

$\mathbb{D} \overset{\text{Conformal}}{\equiv} \mathbb{H}$

$F: \mathbb{H} \to \mathbb{D}$

$F(z) = \dfrac{i - z}{i + z}$

$|F(z)| = \dfrac{|i - z|}{|i + z|} < 1$



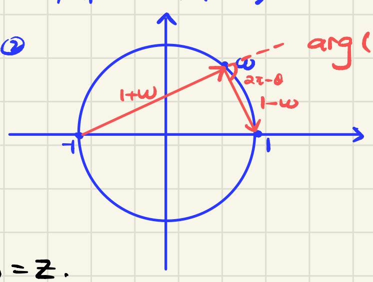

$G: \mathbb{D} \to \mathbb{H}$

$G(\omega) = i \dfrac{1 - \omega}{1 + \omega}$

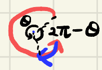
Remark. $z \mapsto \dfrac{az + b}{cz + d}$ fractional linear transformations

① 计算 $\mathrm{Im}(G(\omega))$

②




$\arg\left(\frac{1-\omega}{1+\omega}\right) = \theta$



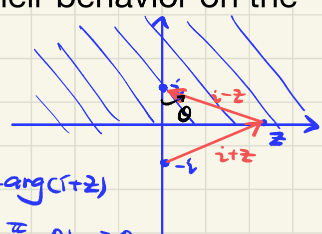

$2\pi - \theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$\leadsto \theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$\Rightarrow \arg\left(i \frac{1-\omega}{1+\omega}\right) = \theta + \frac{\pi}{2} \in (0, \pi)$

And $F \circ G(\omega) = \omega$ $G \circ F(z) = z$.

An interesting aspect of these functions is their behavior on the boundaries of our open sets.




$\partial \mathbb{H}$ = the Real axis $\forall z \in \partial \mathbb{H}$, $|F(z)| = 1$

And $\arg\left(\frac{i - z}{i + z}\right) = \arg(i - z) - \arg(i + z)$

$= \left(\theta + \frac{\pi}{2}\right) - \left(\frac{\pi}{2} - \theta\right) = 2\theta$

$\Rightarrow F(z) = e^{i 2\theta}$ (the same as $z = \tan\theta$) $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

As $\theta$ from $(-\frac{\pi}{2}$ to $\frac{\pi}{2})$
$z$ from $-\infty$ to $+\infty$
$F(z)$ from $(-1$ to $-1)$ (counter-clockwise) $-1$

"and the point $-1$ on the circle corresponds to the "point of infinity" of $\mathbb{H}$."

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More examples (Keep on the boundaries of the sets connected by the functions)

平移: $U=V=\mathbb{C}\ (\mathbb{H})$ $\quad z \longrightarrow z-h$, $\quad w+h \longleftarrow w$ $\quad (h>0,?)$

旋转+放缩 $U=V=\mathbb{C}$ $\quad z \longrightarrow cz$, $\quad c^{-1}w \longleftarrow w$ $\quad (c=re^{i\varphi})$

此时形状变化前后或全等或相似，边界选择 is trivial.

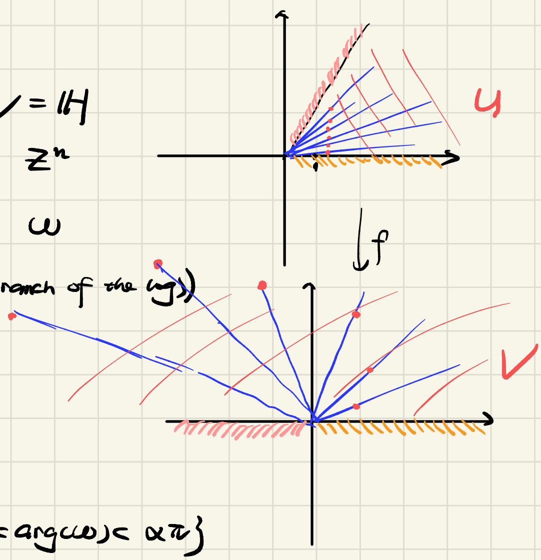
$U=\{z\in\mathbb{C} : 0<\arg(z)<\frac{\pi}{n}\}$ $\qquad V=\mathbb{H}$

$z \longrightarrow z^n$

$w^{\frac{1}{n}} \longleftarrow w$

$(\overset{\shortparallel}{e}{}^{\frac{1}{n}\log w}$ (the principal branch of the log))

不均匀地拉伸切片
(收缩)
(绕原点)

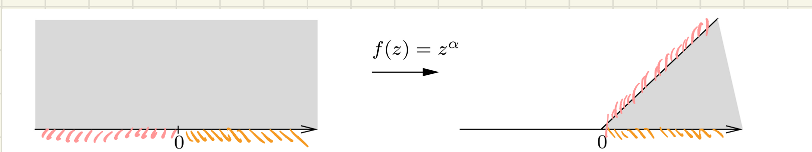


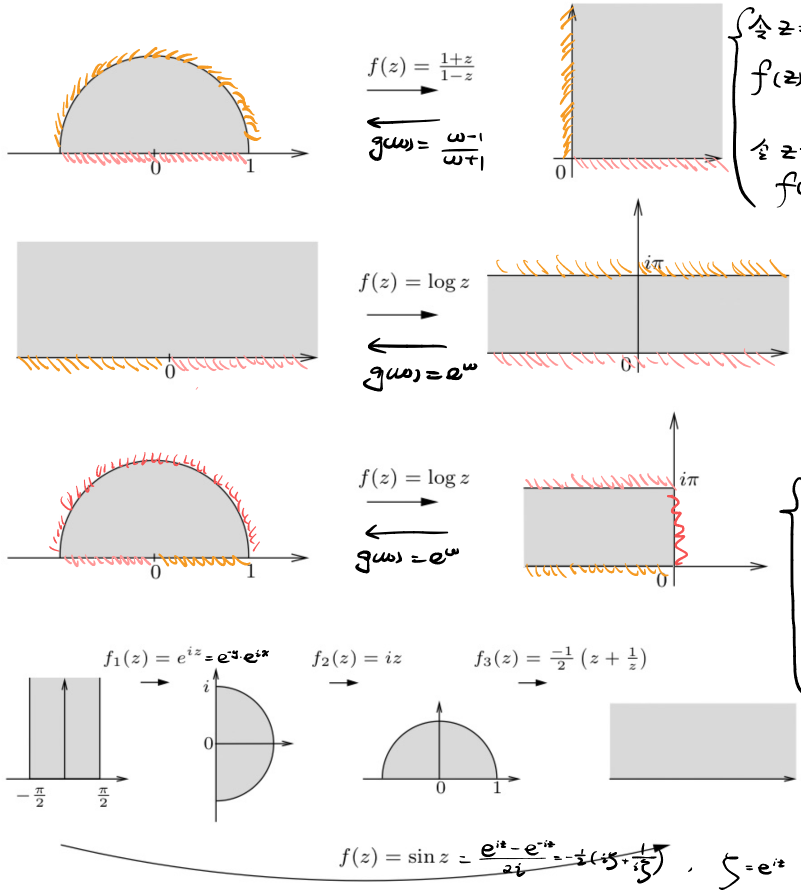

$U=\mathbb{H}$ $\qquad V=\{z\in\mathbb{C} : 0<\arg(w)<\alpha\pi\}$

$f:\ z \longrightarrow z^{\alpha}$

$w^{\frac{1}{\alpha}} \longleftarrow w$

(同理上图的"逆")






令 $z = e^{i\theta}$ $(\theta \in (0,\pi))$

$f(z) = \frac{1+e^{i\theta}}{1-e^{i\theta}} = \frac{e^{-i\frac{\theta}{2}}+e^{i\frac{\theta}{2}}}{e^{-i\frac{\theta}{2}}-e^{i\frac{\theta}{2}}} = \frac{i}{\tan\frac{\theta}{2}}$

令 $z = x \in \mathbb{R}$

$f(z) = \frac{1+x}{1-x}$

$z < 0$ 时

$\log z = \log|z| + i\pi$

$z > 0$ 时

$\log z = \log|z| + i \cdot 0$

$z \in (-1, 0)$

$\log z = \log|z| + i\pi$

$z \in (0, 1)$

$\log z = \log|z| + 0 \cdot i$

$|z| = 1$

$\log z = 0 + i\theta \quad \theta \in (0,\pi)$

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**Dirichlet problem** in the open set $\Omega$, consists of solving $\begin{cases} \Delta u = 0 & \text{in } \Omega \\ u = f & \text{on } \partial\Omega \end{cases}$ and a give $f$ defined on $\Omega$

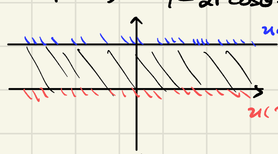
We admit that

when $\Omega$ a unit disc,

$$u(r,\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta-\varphi)u(1,\varphi)\,d\varphi.$$

with $u(1,\varphi) = f(\varphi)$ and $P_r(\theta) = \frac{1-r^2}{1-2r\cos\theta + r^2}$

Question: if $\Omega = \{x+iy : x \in \mathbb{R},\ 0<y<1\}$

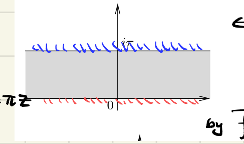



(where $f_i$ continuous and vanish at infinity)

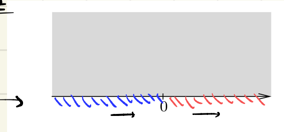
How to construct $u$, or reconstruct $u$ with the unit disc's one?

( $\Omega$ and $\mathbb{D}$ conformally equivalent ?)

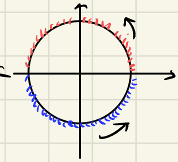
$\Omega \cong$ by $f_1(z)=\pi z$




$\cong$ by $\log z$ / by $f_2(z)=e^z$




$\cong$ unit Disc, $f_3(z)=\frac{i-z}{i+z}$



Thus $G:\Omega\to\mathbb{D}$

$$G(z)=f_3\circ f_2\circ f_1(z)=\frac{i-e^{\pi z}}{i+e^{\pi z}}$$

Conversely we have $F:\mathbb{D}\to\Omega$ $\quad F(\omega)=\frac{1}{\pi}\log\left(i\frac{1-\omega}{1+\omega}\right)$

思路:

$$\begin{tikzcd}
f_0, f_1 \text{ in } \Omega \arrow[r, "``F\text{''}"] \arrow[dr, dashed, "\text{extension}"'] & \tilde{f}_0, \tilde{f}_1 \text{ in } \mathbb{D} \arrow[d, "\text{Possion kernel}"] \\
u(\omega) & \tilde{u}(\omega) \arrow[l, "``G\text{''}"']
\end{tikzcd}$$

"still satisfies $\Delta u(\omega)=0$ ?"

$\Downarrow$

**Lemma 1.3** *Let $V$ and $U$ be open sets in $\mathbb{C}$ and $F:V\to U$ a holomorphic function. If $u:U\to\mathbb{C}$ is a harmonic function, then $u\circ F$ is harmonic on $V$.*

if $u$ harmonic and we know C-R that

if $\exists\ G=u(x,y)+i\,v(x,y)$ holomorphic $\Leftrightarrow \begin{cases}\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}\\ \frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}\end{cases}$

and we could use integral to reconstruct such a $v$.

then $H=G\circ F$ holomorphic

And $H=G(f_1(x,y)+f_2(x,y)\cdot i)$

$=G(f_1(x,y),f_2(x,y))$

$=u(f_1(x,y),f_2(x,y))+i\,v(f_1(x,y),f_2(x,y))$

$=u\circ F(x,y)+i\,v\circ F(x,y)$

thus $u\circ F$ is the real part of $H$ hence harmonic $\square$

"F": take $\omega=e^{i\varphi}$

when $\varphi$ from $-\pi$ to $0$, $F(e^{i\varphi})$ from $i+\infty$ to $i-\infty$

when $\varphi$ from $0$ to $\pi$, $F(e^{i\varphi})$ from $-\infty$ to $\infty$

$$\Rightarrow \tilde{f}(\varphi):=\begin{cases}\tilde{f}_1(\varphi):=f_1(F(e^{i\varphi})-i) & \text{when } -\pi<\varphi<0\\ \tilde{f}_2(\varphi):=f_2(F(e^{i\varphi})) & \text{when } 0<\varphi<\pi\end{cases}$$

and we know $f(0)=f(\pi)=f(-\pi)=0$ (since part)

$\Rightarrow \tilde{f}$ continuous.

$$\Rightarrow \tilde{u}(\omega)=\frac{1}{2\pi}\int_{-\pi}^{\pi}P_r(\theta-\varphi)\tilde{f}(\varphi)\,d\varphi$$

$$=\frac{1}{2\pi}\int_{-\pi}^{0}P_r(\theta-\varphi)\tilde{f}_1(\varphi)\,d\varphi+\frac{1}{2\pi}\int_{0}^{\pi}P_r(\theta-\varphi)\tilde{f}_0(\varphi)\,d\varphi,\ \text{as } \omega=re^{i\theta}$$

Finally let $u(z)=\tilde{u}(G(z))$. and $u$ harmonic. $\square$