Recall that a path connected space is simply connected if the fundamental group is trivial
This is equivalent to any two paths shares the same ends, namely:

$$[e_{x_0}] = [\alpha * \bar{\beta}] \iff [\beta] = [e_{x_0}] * [\beta] = [\alpha * \bar{\beta}] * [\beta] = [\alpha]$$

The key observation is if keep the trace of the contraction process of a loop in a simply connected region into its base point, the trace will form a compact and connected area, therefore locally being covered by small open disc.
We can partition the contraction into small pieces and study it locally. Since Cauchy theorem holds locally in these small discs, this point of view leads us to extend Cauchy theorem in this area. Let's formulate this observation:

Cauchy theorem for simply connected region:

For any nullhomotopic loop h in region Ω with base point x_0

let $H(s,t)$ be the homotopy between h and e_{x_0}

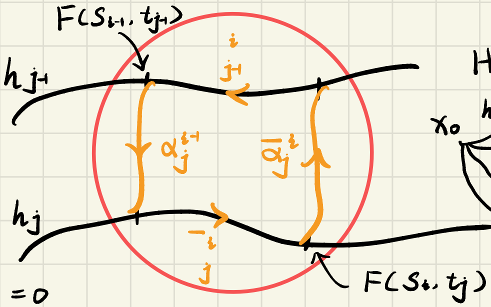
Choose open covering $\mathcal{A} \triangleq \{ B(x, r_x) : x \in H(I^2) \text{ and } B(x, r_x) \text{ inside } \Omega \}$

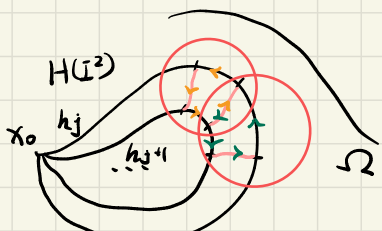
by Lebesgue number

Since $H(I^2)$ compact, there's division $0 = s_0 < s_1 < \cdots < s_n = 1$ $\quad 0 = t_0 < t_1 < \cdots < t_m = 1$

s.t. each $H([s_{i-1}, s_i] \times [t_{j-1}, t_j])$ locates in one ball in $\mathcal{A}$ as pictured

let $\gamma_j^i \triangleq \alpha_j^{i-1} * h_{j+1}^i * \bar{\alpha}_j^i * \bar{h}_j^i$





By Cauchy theorem

for disc : $\int_{\gamma_j^i} f(z)\,dz = 0$

$\square$

Then $\int_{h_j * \bar{h}_{j+1}} f(z)\,dz = 0$ i.e. $\int_{h_j} f(z)\,dz = \int_{h_{j+1}} f(z)\,dz$ hence $\int_h f(z)\,dz = \int_{e_{x_0}} f(z)\,dz = 0$

As an immediate consequence, as implied in the proof the theorem, for holomorphic function f in a simply connected region, integral along any two homotopic paths which sharing the same ends gives the same value. This boils down the ambiguity when defining the primitive of f by path integral, since proceeding results show that this only depends on the final point!