Cauchy Integral Formula

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Enlightening

Observation

If f is holomorphic, then its real and imaginary parts is harmonic that is $\Delta u = \Delta v = 0$

Recall

Integral on a harmonic function along a circle, then the value of integral is determined by the value of function at the center. Moreover, the reversing perspective is even more important.

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Harmonic function

Theorem

$f : \mathbb{R}^2 \to \mathbb{R}$ *with* $\Delta f = 0$
then $\frac{1}{2\pi} \iint_D f(u, v)\, du\, dv = f(x, y)$,
where D is an arbitrary disc centered at (x,y)

A small step further

Question

Is everything holds if the loop winding around any other points in its interior?

Definitely!

The key technique is Cauchy theorem for a keyhole. (Wordplay$\cdots$)

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Cauchy integral formula

Theorem

Suppose f is holomorphic in an open set that contains the closure of a disc D. If C denotes the boundary circle of this disc with the positive orientation, the

$$f(z) = \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{\zeta - z}d\zeta \text{ , where for any point } z \in D$$

Regularity

Observation

The formula enables the differential quotient stabilized. Implicates the regularity of holomorphic function

Theorem

If f is holomorphic in an open set Ω , then f has infinitely many complex derivatives in Ω. Moreover, if $C \subset \Omega$ is a circle whose interior is also contained in Ω, then

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z)^{n+1}} d\zeta$$

for all z in the interior of C.

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1. The Main Theorem
2. Estimation on Derivatives at Infinity
3. Analytic and Rigidity

Somehow Schwartz

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Cauchy inequalities

If f is holomorphic in an open set that contains the closure of a disc D centered at z_0 and of radius R, then

$$\left|f^{(n)}(z_0)\right| \leq \frac{n!\|f\|_C}{R^n},$$

where $\|f\|_C = \sup_{z\in C}|f(z)|$ denotes the supremum of $\|f\|$ on the boundary circle C .

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bounded entire function

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This estimation gives a portrait of a special entire function

Liouville's theorem

If f is entire and bounded, then f is constant.

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Liouville's theorem provides an elegant proof to the fundamental theorem of algebra.

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Every polynomial of degree $n \geq 1$ has precisely n roots in $\mathbb{C}$.

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Implicit power series

Observation

A power series hides in the formula implying holomorphic function is also analytic.

$$\frac{1}{\zeta - z} = \frac{1}{\zeta - z_0 - (z - z_0)} = \frac{1}{\zeta - z_0} \frac{1}{1 - \left(\frac{z - z_0}{\zeta - z_0}\right)}$$

Expansion formula

Suppose f is holomorphic in an open set Ω . If D is a disc centered at z_0 and whose closure is contained in Ω, then f has a power series expansion at z_0

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

for all $z \in D$, and the coefficients are given by

$$a_n = \frac{f^{(n)}(z_0)}{n!} \qquad \text{for all } n \geq 0$$

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Continuation

Remark

Analytic brings rigidity. In other word, the gene of a holomorphic function has been coded in an appropriate open set.
This is a string fact, since open sets in complex plane are rather small, comparing to the case of Zariski topology where open sets are almost the entire space.

Uniqueness of analytic continuation

Suppose f is a holomorphic function in a region Ω that vanishes on a sequence of distinct points with a limit point in Ω. Then f is identically 0.

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