Convention: counting in this section is always with multiplicities.

A basic problem: Given a holomorphic function f on a region, how can one count the number of zeros inside the region?
We may begin with a baby case:

$$f(z) = (z - z_0)^n g(z)$$

It requires a technique to extract the power index n out
Recall the logarithm operation in real analysis will do
Mimic this process and observe that derivation will preserve the desired information of multiplicities

Consider: $\dfrac{f'(z)}{f(z)} = \dfrac{n}{z - z_0} + \dfrac{g'(z)}{g(z)}$ where $g(z)$ nonvanishes near z_0, hence $\dfrac{g'(z)}{g(z)}$ holomorphic

Integral around z_0 shall produce n : $\dfrac{1}{2\pi i}\displaystyle\int_C \dfrac{f'(z)}{f(z)}\,dz = n$

A analogous problem: Given a meromorphic function f on a region, how can one count the number of poles inside the region?
Another baby case:

Similarly, $f(z) = (z - z_0)^{-n} h(z)$

Consider: $\dfrac{f'(z)}{f(z)} = \underbrace{\dfrac{-n}{z - z_0}}_{\text{simple pole}} + \dfrac{h'(z)}{h(z)}$ where $h(z)$ nonvanishes near z_0, hence z_0 isn't a pole for $\dfrac{h'(z)}{h(z)}$

Integral around z_0 shall produce n : $\dfrac{1}{2\pi i}\displaystyle\int_C \dfrac{f'(z)}{f(z)}\,dz = -n$

We combine the two results and formulate them as the well known Argument Principle :

Denote the zeros and poles inside region Ω as $\mathcal{Z}$ and $\mathcal{P}$

then $\#\mathcal{Z} - \#\mathcal{P} = \dfrac{1}{2\pi i}\displaystyle\int_{\partial\Omega} \dfrac{f'(z)}{f(z)}\,dz$ (f novanishes on $\partial\Omega$)

We finish this section by several important applications of the prominent Argument principle

Consider holomorphic function f, having no poles, enforce an appropriate perturbation on f, what happen on the number of zeros? Rouche's theorem gives an approach:

let homotopy $H(z,t) = f(z) + t\,\varepsilon(z)$ for each t, count Z_t

$$n_t \triangleq \# Z_t = \frac{1}{2\pi i}\int_C \frac{\frac{\partial}{\partial z} H(z,t)}{H(z,t)}\,dz \qquad H(z,t) \text{ nonvanishes since } |f(z)| > |\varepsilon(z)|$$

As $C \times I$ compact, $\dfrac{\frac{\partial}{\partial z} H(z,t)}{H(z,t)}$ uniformly continuous on $C \times I$ hence

$n_t = \dfrac{1}{2\pi i}\displaystyle\int_C \dfrac{\frac{\partial}{\partial z} H(z,t)}{H(z,t)}$ is continuous for t, forces n_t constant for it's integer

Invariant of domain is the key characteristic of Euclidean space,
that is continuous function is open map.
There is something similar in complex plane, if we add the
holomorphic condition on the map, namely the open map theorem:

for f holomorphic and nonconstant on Ω, f open.

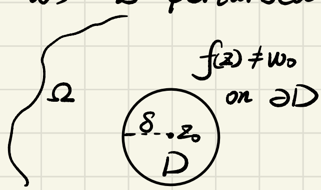
We show that for w near $w_0 = f(z_0)$, $g(z) = f(z) - w$

has zero in disc D centered at z_0, i.e. $w \in f(D)$

It suffice to show that $g(z) = (f(z) - w_0) + (w_0 - w)$ is perturbed

from $f(z) - w_0$, which has a zero z_0

Choose appropriate δ, as pictured, then



$|f(z) - w_0| \geq m > 0$ As long as $|w - w_0| < m$, i.e. serves as

a perturbation, $f(z) - w$ has a root by Rouché's theorem

As a consequence of openness of holomorphic function, for a point in image,
there is always a point in its neighborhood further from origin.
This idea leads to the Maximum Modules Principle:

An interpretation of this idea is that non-constant holomorphic function cannot attain a maximum in the region

As a corollary, if the region is rigged with a compact closure,we can estimate the supremum of the module of image by the periphery of the closure:

$$\sup_{z\in\Omega}|f(z)| \leq \sup_{z\in\bar{\Omega}-\Omega}|f(z)|$$ where f holomorphic on Ω and continuous on $\bar{\Omega}$