

### Preliminaries

• Additive category  $\rightarrow$  Abelian category

• Limit, colimit

• Fibered product / coproduct

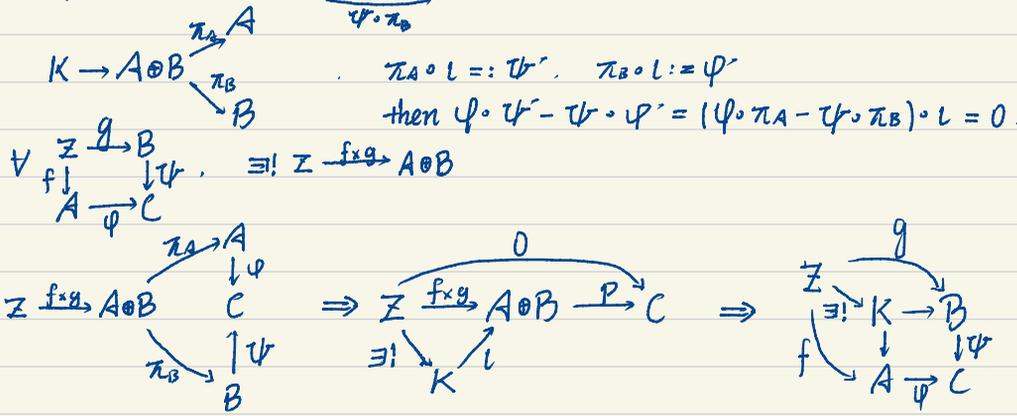
$\mathcal{A}$  an abelian category.  $A, B, C$  objects of  $\mathcal{A}$  with  $A \xrightarrow{\psi} C, B \xrightarrow{\psi} C$

• The fibered product of  $A$  and  $B$  over  $C$  is an object  $A \times_C B$  with  $A \times_C B \xrightarrow{\psi'} B$  and  $A \times_C B \xrightarrow{\psi''} A$  and  $\begin{matrix} A \times_C B & \xrightarrow{\psi'} & B \\ \psi'' \downarrow & \square & \downarrow \psi \\ A & \xrightarrow{\psi} & C \end{matrix}$  and final with this property.

• The fibered coproduct of  $A$  and  $B$  over  $C$  is an object  $A \sqcup_C B$  with  $C \xrightarrow{\psi} B$  and  $C \xrightarrow{\psi'} A$  and  $\begin{matrix} C & \xrightarrow{\psi} & B \\ \psi' \downarrow & \square & \downarrow \psi \\ A & \xrightarrow{\psi'} & A \sqcup_C B \end{matrix}$  and initial with this property.

Claim: Fibered products / coproducts exist in abelian categories.

[Proof]: consider  $A \oplus B \xrightarrow{\psi} C, \psi = \psi' \circ \pi_A - \psi'' \circ \pi_B, K \triangleq \ker \psi$  the kernel of  $\psi$



By definitions.  $A \times_C B = K. \square$

Lemma: For  $A \times_C B \xrightarrow{\psi'} B$   $\ker \psi = \ker \psi'$   
 $\psi'' \downarrow \square \downarrow \psi$   $\ker \psi = \ker \psi''$

For  $C \xrightarrow{\psi} B$   $\operatorname{coker} \psi = \operatorname{coker} \psi'$   
 $\psi' \downarrow \square \downarrow \psi$   $\operatorname{coker} \psi = \operatorname{coker} \psi''$

Lemma:  $A \times_C B \xrightarrow{\psi'} B \Rightarrow \psi'$  is an epimorphism  
 $\psi'' \downarrow \square \downarrow \psi$   $\Rightarrow \psi''$  is also an epimorphism.)

$C \xrightarrow{\psi} B \Rightarrow \psi$  is a monomorphism  
 $\psi' \downarrow \square \downarrow \psi$   $\Rightarrow \psi'$  is also a monomorphism.)



$$A := \varinjlim \mathcal{A}$$

Why the colimits?

→ Large enough to contain arbitrary  $I \cong A$

The colimit can be constructed by joining pointed sets at the distinguished point and then taking a quotient. Details similar with the construction of  $\varinjlim A_i$  in  $\mathcal{R}$ -mod. (i.e. pinning all  $\text{Hom}_{\mathcal{A}}(I, A)$  together at the zero morphisms).

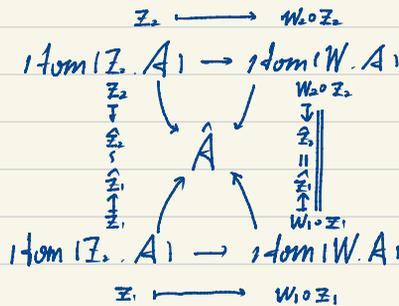
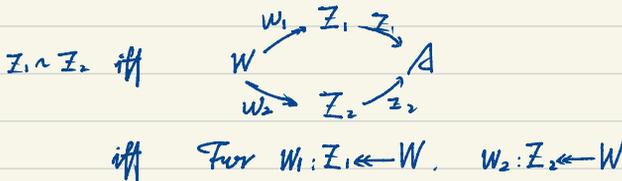
Now let's combine these ideas.

→ here is why we need  $\mathcal{A}$  to be small.

For  $I \cong A$  "element of  $\mathcal{A}$ ",  $I \in \text{Hom}_{\mathcal{A}}(I, A)$ .  $\text{Hom}_{\mathcal{A}}(I, A) \rightarrow \varinjlim \mathcal{A}$ ,  $I \rightarrow \hat{I}$

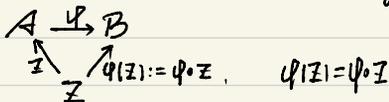
Claim:  $I_1 \sim I_2$  iff  $\hat{I}_1 = \hat{I}_2$ .

[Proof]:



iff  $\hat{I}_1 = \hat{I}_2$   $\square$

As a result, recall that in the beginning we define  $\varphi(I)$  as follow:



this induces  $\hat{\varphi}: \hat{A} \rightarrow \hat{B}$ ,  $\hat{I} \mapsto \hat{\varphi}(\hat{I}) := \varphi \circ z$ , correspondingly — We successfully find the set-theoretic correspondence of objects in  $\mathcal{A}$  and their behavior.

The next question is, in what details the correspondence can describe the further in  $\mathcal{A}$ ?

Lemma 1  $I \sim 0$  iff  $I = 0$ . Further,  $\varphi: A \rightarrow B$ ,  $\varphi = 0$  iff  $\hat{\varphi}(\hat{I}) = 0, \forall \hat{I} \in \hat{A}$ .

[Proof]:  $z \sim 0 \iff$

$$\begin{array}{ccc}
 W & \xrightarrow{w_1} & Z \xrightarrow{z} A \\
 & \xrightarrow{w_2} & Z \xrightarrow{0} A
 \end{array}
 \quad z \circ w_1 = 0 \circ w_2 = 0 \quad \iff \quad z = 0$$

$$\varphi = 0 \Rightarrow \forall \hat{z} \in \hat{A}, \hat{z} \xrightarrow{z} A, \hat{\varphi}(\hat{z}) = \widehat{\varphi \circ z} = 0$$

conversely,  $\forall \hat{z} \in \hat{A}, \hat{z} \xrightarrow{z} A, \hat{\varphi}(\hat{z}) = 0 \Rightarrow A \xrightarrow{id_A} A, \varphi \circ id_A \sim 0$   
 $\Rightarrow \varphi \circ id_A = \varphi = 0 \quad \square$

Lemma 2.  $\varphi$  is a monomorphism iff  $\hat{\varphi}$  is injective.  
 $\varphi$  is an epimorphism iff  $\hat{\varphi}$  is surjective.

[Proof]:

$\varphi$  is an epimorphism  $\Rightarrow \forall \hat{z} \in \hat{B}$  i.e.  $\forall z: Z \rightarrow B$

$$\begin{array}{ccc}
 A \times_B Z & \xrightarrow{\varphi'} & Z \\
 \downarrow z' & \square & \downarrow z \\
 A & \xrightarrow{\varphi} & B
 \end{array}
 \quad \varphi' \text{ is an epimorphism}$$

Consider  $A \times_C Z \xrightarrow{id} A \times_B Z \xrightarrow{\varphi \circ z} B \Rightarrow z \sim \varphi \circ z'$  i.e.  $\hat{\varphi}(\hat{z}') = \widehat{\varphi \circ z'} = \hat{z}$   
 hence  $\hat{\varphi}$  is surjective.

Conversely,  $\hat{\varphi}$  is surjective  $\Rightarrow$  For  $id_B: B \rightarrow B \exists z: Z \rightarrow A$  s.t.  $\hat{\varphi}(\hat{z}) = id_B$

$$\begin{array}{ccc}
 \Rightarrow \varphi \circ z \sim id_B \Rightarrow A \xrightarrow{\varphi} B & \Rightarrow \varphi \circ z \circ w_1 = id_B \circ w_2 = w_2 \text{ is an epimorphism} \\
 \uparrow z & \parallel id_B & \Rightarrow \varphi \text{ is an epimorphism.} \\
 Z \xleftarrow{w_1} W \xrightarrow{w_2} B & &
 \end{array}$$

$\varphi$  is a monomorphism  $\Rightarrow (\forall z_1: Z_1 \rightarrow A, z_2: Z_2 \rightarrow A, \varphi(z_1) \sim \varphi(z_2))$  i.e.

$$\begin{array}{ccc}
 W & \xrightarrow{w_1} & Z_1 \xrightarrow{z_1} A \xrightarrow{\varphi} B \\
 & \xrightarrow{w_2} & Z_2 \xrightarrow{z_2} A \xrightarrow{\varphi} B
 \end{array}$$

$$\varphi \circ z_1 \circ w_1 = \varphi \circ z_2 \circ w_2 \Rightarrow z_1 \circ w_1 = z_2 \circ w_2 \quad \text{i.e. } z_1 \sim z_2$$

$\Rightarrow \hat{\varphi}$  is injective.

Conversely,  $\hat{\varphi}$  is injective  $\Rightarrow (\forall z: Z \rightarrow A, \hat{\varphi}(\hat{z}) = 0$  i.e.  $\varphi \circ z \sim 0 \Rightarrow z = 0$ )

$$\Rightarrow (\forall z: Z \rightarrow A, \varphi \circ z = 0 \Rightarrow z = 0)$$

$\Rightarrow \varphi$  is a monomorphism  $\square$ .

Lemma 3. Suppose  $i: K \rightarrow A$ ,  $j: I \rightarrow B$  are the kernel and the image of  $\varphi$  respectively.

then  $i: \hat{K} \xrightarrow{\cong} \hat{\varphi}^{-1}(0)$ ,  $j: \hat{I} \xrightarrow{\cong} \hat{\varphi}(\hat{A})$  (i.e. the functor  $\mathcal{A} \rightarrow \text{Set}^*$  let kernels and images in abelian categories may be identified with those defined in general set-theoretic language.)

[Proof]:

It suffices to show  $i(\hat{I}) = \hat{\varphi}^{-1}(0)$ ,  $j(\hat{I}) = \hat{\varphi}(\hat{A})$ , for  $i, j$  are monomorphisms,  $i, j$  are injective by lemma 2.

i) Firstly we show that  $j(\hat{I}) \subseteq \hat{\varphi}(\hat{A})$ .

in fact,  $\forall z_1: Z_1 \rightarrow I$ .

$$\begin{array}{ccc} & \varphi & \\ & \curvearrowright & \\ A & \xrightarrow{\quad} & I \xrightarrow{j} B \\ \uparrow z_1 \square \uparrow z_1/j(z_1) & & \\ A \times_I Z_1 & \rightarrow & Z_1 \end{array}$$

$$\Rightarrow j \circ z_1 \wedge \varphi \circ z_1, \text{ i.e. } j(\hat{z}_1) = \hat{\varphi}(\hat{z}_1) \in \hat{\varphi}(\hat{A})$$

Then we show that  $\hat{\varphi}(\hat{A}) \subseteq j(\hat{I})$ .

$\forall z_1: Z_1 \rightarrow A$ ,

$$\begin{array}{ccc} & \varphi & \\ & \curvearrowright & \\ A & \xrightarrow{\pi} & I \xrightarrow{j} B \\ \uparrow z_1 \nearrow \pi \circ z_1 & & \\ Z_1 & & \end{array}$$

$$\Rightarrow \varphi \circ z_1 = j \circ \pi \circ z_1$$

$$\Rightarrow \varphi \circ z_1 \wedge j \circ \pi \circ z_1$$

$$\Rightarrow \hat{\varphi}(\hat{z}_1) = \hat{j}(\hat{\pi} \circ z_1) \in j(\hat{I})$$

iii) First,  $i(\hat{K}) \subseteq \hat{\varphi}^{-1}(0)$ , i.e.  $\hat{\varphi} \circ i(\hat{K}) = 0$ . This is trivial:

$\forall z_1: Z_1 \rightarrow K$ ,

$$\begin{array}{ccc} & 0 & \\ & \curvearrowright & \\ K & \xrightarrow{\quad} & A \xrightarrow{\varphi} B \\ \uparrow z_1 \nearrow l \circ z_1 & & \\ Z_1 & & \end{array}$$

$$\varphi \circ l \circ z_1 = 0 \circ z_1 = 0$$

Then  $\hat{\varphi}^{-1}(0) \subseteq i(\hat{K})$

$\forall \hat{z}_2 \in \hat{\varphi}^{-1}(0)$ , i.e.  $z_2: Z_2 \rightarrow A$ ,  $\varphi \circ z_2 = 0$ . By the definition of kernel:

$$\begin{array}{ccc} K & \xrightarrow{\quad} & A \xrightarrow{\varphi} B \\ \exists! \hat{z}_2 \nearrow \uparrow z_2 & & \nearrow 0 \\ Z_2 & & \end{array}$$

$$z_2 = l \circ \hat{z}_2 \Rightarrow \hat{z}_2 = \hat{i}(\hat{z}_2) \quad \square$$

If we define a sequence  $S_1 \xrightarrow{f_1} S_2 \xrightarrow{f_2} \dots \xrightarrow{f_{n-1}} S_n \xrightarrow{f_n} S_{n+1}$  in  $\text{Set}^*$  is exact at  $S_n$  if  $f_{n-1}(S_{n-1}) = f_n^{-1}(0)$  (which coincides with our set-theoretic recognition in R-mod). As a result, we have the following corollary immediately

Prop: A sequence  $A \xrightarrow{f} B \xrightarrow{g} C$  in  $\mathcal{A}$  iff  $\hat{A} \xrightarrow{\hat{f}} \hat{B} \xrightarrow{\hat{g}} \hat{C}$  is exact in  $\text{Set}^*$ .

Till now, we have shown that we may consider the objects of  $\mathcal{A}$  as sets, and morphisms as maps between sets with no loss of generality. In fact, we may correspond objects and morphisms in  $\mathcal{A}$  with those in  $\text{Set}^*$ , and this correspondence is ensured not changing the essential properties of kernels, cokernels, zeros, equivalence relations, exactness and so on. And in fact, we do only care about the behavior. When we are about to handle a diagram in  $\mathcal{A}$ , we can check the commutativity and exactness of the diagram by element-chasing.

However, though may be not so necessary, we may refine this conclusion more elegantly. In fact, the functor  $F: \mathcal{A} \rightarrow \text{Set}^*$  defined above is neither faithful nor full. (This functor is after all a correspondence between objects and sets. It is useful enough, though, not elegant enough.) Here, a miracle theorem occurs:

Thm (Freyd-Mitchell):  $\mathcal{A}$  a small abelian category. Then there exists a fully faithful, exact functor  $\mathcal{A} \rightarrow \mathcal{R}\text{-mod}$  for some ring  $\mathcal{R}$ . (i.e. an "embedding").

Remark:  $\mathcal{R}$  is not necessarily commutative.