Premilinaries

- Additive category $\longrightarrow$ Abelian category
- Limit. colimit
- Fibered product / coproduct.

$\mathcal{A}$ an abelian category. $A. B. C$ objects of $\mathcal{A}$ with $A \xrightarrow{\varphi} C$. $B \xrightarrow{\psi} C$

• The fibered product of A and B over C is an object $A\times_C B$ with
$$\begin{tikzcd} A\times_C B \arrow[r, "\varphi'"] \arrow[d, "\psi'"'] \arrow[dr, phantom, "\square"] & B \arrow[d, "\psi"] \\ A \arrow[r, "\varphi"] & C \end{tikzcd}$$
and final with this property.

• The fibered coproduct of A and B over C is an object $A\amalg_C B$ with
$$\begin{tikzcd} C \arrow[r, "\psi"] \arrow[d, "\varphi"'] & B \arrow[d, "\varphi'"] \\ A \arrow[r, "\psi'"] & A\amalg_C B \end{tikzcd}$$
and initial with this property.

Claim: Fibered products / coproducts exist in abelian categories.

[Proof]: consider
$$\begin{tikzcd} A\oplus B \arrow[r, bend left, "\varphi\circ\pi_A"] \arrow[r, bend right, "\psi\circ\pi_B"'] & C \end{tikzcd}.$$
$p := \varphi\circ\pi_A - \psi\circ\pi_B$. $K \xrightarrow{\iota} A\oplus B$ the kernel of p

$$\begin{tikzcd} & & A \\ K \arrow[r] & A\oplus B \arrow[ur, "\pi_A"] \arrow[dr, "\pi_B"'] & \\ & & B \end{tikzcd}$$

$\pi_A\circ\iota =: \psi'$, $\pi_B\circ\iota := \varphi'$

then $\varphi\circ\psi' - \psi\circ\varphi' = (\varphi\circ\pi_A - \psi\circ\pi_B)\circ\iota = 0$.

$$\forall \begin{tikzcd} Z \arrow[r, "g"] \arrow[d, "f"'] & B \arrow[d, "\psi"] \\ A \arrow[r, "\varphi"'] & C \end{tikzcd}, \quad \exists!\ Z \xrightarrow{f\times g} A\oplus B$$

$$\begin{tikzcd} & & A \arrow[d, "\varphi"] \\ Z \arrow[r, "f\times g"] & A\oplus B \arrow[ur, "\pi_A"] \arrow[dr, "\pi_B"'] & C \\ & & B \arrow[u, "\psi"'] \end{tikzcd} \Rightarrow \begin{tikzcd} Z \arrow[r, "f\times g"] \arrow[rr, bend left, "0"] \arrow[dr, "\exists!"'] & A\oplus B \arrow[r, "p"] & C \\ & K \arrow[u, "\iota"'] & \end{tikzcd} \Rightarrow \begin{tikzcd} Z \arrow[dr, "\exists!"] \arrow[drr, bend left, "g"] \arrow[ddr, bend right, "f"'] & & \\ & K \arrow[r] \arrow[d] & B \arrow[d, "\psi"] \\ & A \arrow[r, "\varphi"'] & C \end{tikzcd}$$

By definitions. $A\times_C B = K$. □

Lemma: For
$$\begin{tikzcd} A\times_C B \arrow[r, "\varphi'"] \arrow[d, "\psi'"'] \arrow[dr, phantom, "\square"] & B \arrow[d, "\psi"] \\ A \arrow[r, "\varphi"] & C \end{tikzcd},$$
$\ker\varphi = \ker\varphi'$.
$\ker\psi = \ker\psi'$

For
$$\begin{tikzcd} C \arrow[r, "\psi"] \arrow[d, "\varphi"'] \arrow[dr, phantom, "\square"] & B \arrow[d, "\varphi'"] \\ A \arrow[r, "\psi'"] & A\amalg_C B \end{tikzcd},$$
$\operatorname{coker}\varphi = \operatorname{coker}\varphi'$.
$\operatorname{coker}\psi = \operatorname{coker}\psi'$.

Lemma:
$$\begin{tikzcd} A\times_C B \arrow[r, "\varphi'"] \arrow[d, "\psi'"'] \arrow[dr, phantom, "\square"] & B \arrow[d, "\psi"] \\ A \arrow[r, "\varphi"] & C \end{tikzcd} \Rightarrow$$
(φ is an epimorphism
$\Rightarrow \varphi'$ is also an epimorphism.)

$$\begin{tikzcd} C \arrow[r, "\varphi"] \arrow[d, "\psi"'] & B \arrow[d, "\psi'"] \\ A \arrow[r, "\varphi'"'] & A\amalg_C B \end{tikzcd} \Rightarrow$$
(φ is a monomorphism
$\Rightarrow \varphi'$ is also a monomorphism).

Lemma: $\bullet$
$$\begin{tikzcd} D \arrow[r, "\varphi'"] \arrow[d, "\psi'"'] & A \arrow[d, "\psi"] \\ B \arrow[r, "\varphi"] & C \end{tikzcd}$$

$\bullet$ $0 \to D \to A \oplus B \to C$ exact iff $D \cong A \times_C B$

$D \to A \oplus B \to C \to 0$ exact iff $C \cong A \amalg_D B$

Identify objects with sets : $\mathcal{A} \to Set^*$. (the category of pointed sets.)

Idea 1.: we want to define an element $z \in A$ as $Z \xrightarrow{z} A$ for an object $Z \in Obj(\mathcal{A})$.

For a morphism $A \xrightarrow{\varphi} B$, $z \mapsto \varphi(z)$, we consider
$$\begin{tikzcd} A \arrow[r, "\varphi"] & B \\ & Z \arrow[ul, "z"] \arrow[u, "\varphi(z) := \varphi \circ z"'] \end{tikzcd}$$
we only care about the behavior.

[However, this simple consider does not work well all the time.]

stipulate that :

$z_1 : Z_1 \to A$, $z_2 : Z_2 \to A$ determine the same "element" if $\exists W \in Obj(\mathcal{A})$. s.t.
$$\begin{tikzcd} & Z_1 \arrow[dr, "z_1"] & \\ W \arrow[ur, "w_1"] \arrow[dr, "w_2"'] & & A \\ & Z_2 \arrow[ur, "z_2"'] & \end{tikzcd}$$
commute. (denoted by $z_1 \sim z_2$)

$$\begin{tikzcd} & & Z_1 \arrow[dr, "z_1"] & \\ & W_1 \arrow[ur, "w_1"] \arrow[dr, "w_2"'] & & A \\ & & Z_2 \arrow[ur, "z_2"'] & \\ W_1 \oplus W_2 \arrow[uur] \arrow[ddr] & & & \\ & & Z_2 \arrow[dr, "z_2"] & \\ & W_2 \arrow[ur, "w_3"] \arrow[dr, "w_4"'] & & A \\ & & Z_3 \arrow[ur, "z_3"'] & \end{tikzcd}$$

$\Rightarrow$ The stipulation above is indeed an equivalence relation.

Question : Since we consider elements as morphisms from some sets, what is the corresponding set? $\Rightarrow$ The sets of morphisms, say, $Hom_{\mathcal{A}}(Z, A)$.

Idea 2 : The set corresponding to A

$\mathcal{A}$ a small abelian category.

$\mathcal{A}_{\twoheadrightarrow}$: $Obj(\mathcal{A}_{\twoheadrightarrow}) = Obj(\mathcal{A})$.

$Hom_{\mathcal{A}_{\twoheadrightarrow}}(Z, W) = \{\varphi \in Hom_{\mathcal{A}}(W, Z) \mid \varphi : W \twoheadrightarrow Z\}$

For $A \in Obj(\mathcal{A}_{\twoheadrightarrow})$, $\mathcal{H}_A : \mathcal{A}_{\twoheadrightarrow} \to Set^*$.

$\mathcal{H}_A(Z) = Hom_{\mathcal{A}}(Z, A)$.

$\alpha : B \to C$. (in fact, $C \xrightarrow{\alpha} B$ in $\mathcal{A}$) $\quad \mathcal{H}_A(\alpha) : Hom_{\mathcal{A}}(B, A) \to Hom_{\mathcal{A}}(C, A)$, $(B \xrightarrow{\varphi} A) \mapsto (C \xrightarrow{\alpha} B \xrightarrow{\varphi} A)$

$\hat{A} := \varinjlim h_A$.   Why the colimits?

⇒ Large enough to contain arbitrary $Z \xrightarrow{z} A$

The colimit can be constructed by joining pointed sets at the distinguished point and then taking a quotient. Details similar with the construction of $\varinjlim_i A_i$ in R-mod. (i.e. pinning all $Hom_{\mathcal{A}}(\cdot, A)$ together at the zero morphisms).

Now let's combine these ideas. Here is why we need $\mathcal{A}$ to be small.

For $Z \xrightarrow{z} A$ "element of A", $z \in Hom_{\mathcal{A}}(Z, A)$. $Hom_{\mathcal{A}}(Z, A) \to \varinjlim h_A$, $z \to \hat{z}$

Claim: $z_1 \sim z_2$ iff $\hat{z}_1 = \hat{z}_2$.

[Proof]:

$z_1 \sim z_2$ iff
$$\begin{tikzcd} & Z_1 \arrow[dr, "z_1"] & \\ W \arrow[ur, "w_1"] \arrow[dr, "w_2"'] & & A \\ & Z_2 \arrow[ur, "z_2"'] & \end{tikzcd}$$

iff For $w_1 : Z_1 \twoheadleftarrow W$, $w_2 : Z_2 \twoheadleftarrow W$

$$z_2 \longmapsto w_2 \circ z_2$$
$$\begin{tikzcd} Hom(Z_2, A) \arrow[rr] \arrow[dr] & & Hom(W, A) \arrow[dl] \\ & \hat{A} & \\ Hom(Z_2, A) \arrow[rr] \arrow[ur] & & Hom(W, A) \arrow[ul] \end{tikzcd}$$
$$z_1 \longmapsto w_1 \circ z_1$$

$$\begin{array}{ccc} z_2 & & w_2 \circ z_2 \\ \downarrow & & \downarrow \\ \hat{z}_2 & & \hat{z}_2 \\ \wr & & \| \\ \hat{z}_1 & & \hat{z}_1 \\ \uparrow & & \uparrow \\ z_1 & & w_1 \circ z_1 \end{array}$$

iff $\hat{z}_1 = \hat{z}_2$ $\square$

As a result, recall that in the beginning we define $\varphi(z)$ as follow:

$$\begin{tikzcd} A \arrow[r, "\varphi"] & B \\ Z \arrow[u, "z"] \arrow[ur, "\varphi(z) := \varphi \circ z"'] & \end{tikzcd}$$
$\varphi(z) = \varphi \circ z$

this induces $\hat{\varphi} : \hat{A} \to \hat{B}$, $\hat{z} \mapsto \hat{\varphi}(\hat{z}) := \varphi \circ z$, correspondingly — We successfully find the set-theoretic correspondence of objects in $\mathcal{A}$ and their behavior.

The next question is, in what details the correspondence can describe the feather in $\mathcal{A}$?

Lemma 1 $z \sim 0$ iff $z = 0$. Further, $\varphi : A \to B$, $\varphi = 0$ iff $\hat{\varphi}(\hat{z}) = 0$, $\forall \hat{z} \in \hat{A}$.

[Proof]: $z \sim 0 \iff$
$$\begin{tikzcd} & Z \arrow[dr, "z"] & \\ W \arrow[ur, "w_1"] \arrow[dr, "w_2"'] & & A \\ & Z \arrow[ur, "0"'] & \end{tikzcd}$$
$z\circ w_1 = 0\circ w_2 = 0 \iff z=0$

$\varphi = 0 \Rightarrow \forall \hat{z}\in\hat{A}.\ Z\xrightarrow{z}A.\ \hat{\varphi}(\hat{z}) = \widehat{\varphi\circ z} = 0$

conversely. $\forall \hat{z}\in\hat{A}.\ Z\xrightarrow{z}A\quad \hat{\varphi}(\hat{z}) = 0 \Rightarrow A\xrightarrow{id_A}A.\quad \varphi\circ id_A \sim 0$

$\Rightarrow \varphi\circ id_A = \varphi = 0$ □

Lemma 2. φ is a monomorphism iff $\hat{\varphi}$ is injective.

φ is an epimorphism iff $\hat{\varphi}$ is surjective.

[Proof]:

φ is an epimorphism $\Rightarrow \forall \hat{z}\in\hat{B}.$ i.e. $\forall z: Z\to B$.

$$\begin{tikzcd} A\times_B Z \arrow[r, "\varphi'"] \arrow[d, "z'"'] \arrow[dr, phantom, "\square"] & Z \arrow[d, "z"] \\ A \arrow[r, "\varphi"'] & B \end{tikzcd}$$

φ' is an epimorphism

Consider
$$\begin{tikzcd} & A\times_c Z \arrow[dr, bend left, "\varphi\circ z'"] & \\ A\times_c Z \arrow[ur, bend left, "id"] \arrow[dr, bend right, "\varphi'"'] & & B \\ & Z \arrow[ur, bend right, "z"'] & \end{tikzcd}$$
$\Rightarrow z\sim\varphi\circ z'$. i.e. $\hat{\varphi}(\hat{z'}) = \widehat{\varphi\circ z'} = \hat{z}$.

Hence $\hat{\varphi}$ is surjective.

Conversely. $\hat{\varphi}$ is surjective. $\Rightarrow$ For $id_B: B\to B$. $\exists z: Z\to A$, s.t. $\hat{\varphi}(\hat{z}) = \widehat{id_B}$.

$\Rightarrow \varphi\circ z \sim id_B \Rightarrow$
$$\begin{tikzcd} A \arrow[rr, "\varphi"] & & B \\ Z \arrow[u, "z"] & W \arrow[l, two heads, "w_1"'] \arrow[r, two heads, "w_2"] & B \arrow[u, equal, "id_B"'] \end{tikzcd}$$
$\Rightarrow \varphi\circ z\circ w_1 = id_B\circ w_2 = w_2$ is an epimorphism

$\Rightarrow \varphi$ is an epimorphism.

φ is a monomorphism $\Rightarrow (\forall z_1: Z_1\to A.\ z_2: Z_2\to A.\ \varphi(z_1)\sim\varphi(z_2)$, i.e.
$$\begin{tikzcd} & Z_1 \arrow[dr, dashed, "z_1"] \arrow[drr, bend left, "\varphi\circ z_1"] & & \\ W \arrow[ur, "w_1"] \arrow[dr, "w_2"'] & & A \arrow[r, "\varphi"] & B \\ & Z_2 \arrow[ur, dashed, "z_2"'] \arrow[urr, bend right, "\varphi\circ z_2"'] & & \end{tikzcd}$$
$\varphi\circ z_1\circ w_1 = \varphi\circ z_2\circ w_2 \Rightarrow z_1\circ w_1 = z_2\circ w_2$. i.e. $z_1\sim z_2)$

$\Rightarrow \hat{\varphi}$ is injective.

Conversely. $\hat{\varphi}$ is injective $\Rightarrow (\forall z: Z\to A.\ \hat{\varphi}(\hat{z}) = 0.$ i.e. $\varphi\circ z\sim 0 \Rightarrow z = 0)$

$\Rightarrow (\forall z: Z\to A.\ \varphi\circ z = 0 \Rightarrow z = 0)$

$\Rightarrow \varphi$ is a monomorphism □.

Lemma 3. Suppose $\iota: K\hookrightarrow A$, $j: I\to B$ are the kernel and the image of φ respectively.

then $\hat{i}: \hat{K} \xrightarrow{\sim} \hat{\varphi}^{-1}(0)$, $\hat{j}: \hat{I} \xrightarrow{\sim} \hat{\varphi}(\hat{A})$ (i.e. the functor $\mathcal{A} \to Set^*$ let kernels and images in abelian categories may be identified with those defined in general set-theoretic language.)

[Proof]:

It suffices to show $\hat{i}(\hat{I}) = \hat{\varphi}^{-1}(0)$, $\hat{j}(\hat{I}) = \varphi(\hat{A})$, for i, j are monomorphisms, $\hat{i}$, $\hat{j}$ are injective by lemma 2.

i) Firstly we show that $\hat{j}(\hat{I}) \le \hat{\varphi}(\hat{A})$.

in fact, $\forall z_1 : Z_1 \to I$.

$$\begin{tikzcd} A \arrow[r, two heads] \arrow[rr, bend left, "\varphi"] & I \arrow[r, hook, "j"] & B \\ A \times_I Z \arrow[u, "z_1'"] \arrow[r] \arrow[ur, phantom, "\square"] & Z_1 \arrow[u, "z_1"] \arrow[ur, "j(z)"'] & \end{tikzcd} \quad \Rightarrow \quad j \circ z_1 \sim \varphi \circ z_1', \text{ i.e. } \hat{j}(\hat{z}_1) = \hat{\varphi}(\hat{z}_1') \le \hat{\varphi}(\hat{A}).$$

Then we show that $\hat{\varphi}(\hat{A}) \le \hat{j}(I)$.

$\forall z : Z_2 \to A$,

$$\begin{tikzcd} A \arrow[r, two heads, "\pi"] \arrow[rr, bend left, "\varphi"] & I \arrow[r, hook, "j"] & B \\ Z_2 \arrow[u, "z_2"] \arrow[ur, "\pi \circ z_2"'] & & \end{tikzcd}$$

$\Rightarrow \varphi \circ z_2 = j \circ \pi \circ z_2$

$\Rightarrow \varphi \circ z_2 \sim j \circ \pi \circ z_2$

$\Rightarrow \hat{\varphi}(\hat{z}_2) = \hat{j}(\widehat{\pi \circ z_2}) \le \hat{j}(\hat{I})$.

ii) First, $\hat{i}(\hat{K}) \le \hat{\varphi}^{-1}(0)$, i.e. $\hat{\varphi} \circ \hat{i}(\hat{K}) = 0$. This is trivial:

$\forall z_1 : Z_1 \to K$.

$$\begin{tikzcd} K \arrow[r, hook, "\iota"] \arrow[rr, bend left, "0"] & A \arrow[r, "\varphi"] & B \\ Z_1 \arrow[u, "z_1"] \arrow[ur, "\iota \circ z_1"'] & & \end{tikzcd} \qquad \varphi \circ \iota \circ z_1 = 0 \circ z_1 = 0$$

Then $\hat{\varphi}^{-1}(0) \le \hat{i}(K)$

$\forall \hat{z}_2 \in \hat{\varphi}^{-1}(0)$, i.e. $z_2 : Z_2 \to A$, $\varphi \circ z_2 = 0$. By the definition of kernel:

$$\begin{tikzcd} K \arrow[r, hook, "\iota"] & A \arrow[r, "\varphi"] & B \\ & Z_2 \arrow[ul, dashed, "\exists ! \hat{z}_2"] \arrow[u, "z_2"] \arrow[ur, "0"'] & \end{tikzcd} \qquad z_2 = \iota \circ \hat{z}_2 \Rightarrow \hat{z}_2 = \hat{i}(\hat{\hat{z}}_2) \quad \square$$

If we define a sequence $S_1 \xrightarrow{f_1} S_2 \xrightarrow{f_2} \cdots \to S_{n-1} \xrightarrow{f_{n-1}} S_n \xrightarrow{f_n} S_{n+1}$ in Set^* is exact at S_n if $f_{n-1}(S_{n-1}) = f_n^{-1}(0)$ (which coincides with our set-theoretic regconition in R-mod). As a result, we have the following corollary immediately

Prop: A sequence $A \xrightarrow{\varphi} B \xrightarrow{\psi} C$ in $\mathcal{A}$ iff $\hat{A} \xrightarrow{\hat{\varphi}} \hat{B} \xrightarrow{\hat{\psi}} \hat{C}$ is exact in Set^*.

Till now, we have shown that we may considered the objects of $\mathcal{A}$ as sets, and morphisms as maps between sets with no loss of generality. In fact, we may correspond objects and morphisms in $\mathcal{A}$ with those in Set^*, and this correspondence is ensured not changing the essential properties of kernels, cokernels, zeros, equivalent relations, exactness and so on. And in fact, we do only care about the behavior. When we are about to handle a diagram in $\mathcal{A}$, we can check the commutativity and exactness of the diagram by element-chasing.

However, though may be not so necessary, we may refine this conclusion more elegant. In fact, the functor $F: \mathcal{A} \to Set^*$ defined above is neither faithful nor full. (This functor is after all a correspondence between objects and sets. It is useful enough, though, not elegant enough.) Here, a miracle theorem occurs:

Thm (Freyd-Mitchell): $\mathcal{A}$ a small abelian category. Then there exists a fully faithful, exact functor $\mathcal{A} \to R\text{-mod}$ for some ring R. (i.e. an "imbedding").

Remark: R is not necessary commutative.