From Calculus to Cohomology via Differential Forms

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Outline

Motivation: distinguish spaces

Goal: study the genuine shapes of spaces and distinguish them (Here we focus on the case of open sets in $\mathbb{R}^n$)

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Example

If two spaces have different numbers of connected components, then they must be genuinely different. Let $U \subset \mathbb{R}^n$ be an open subset,

$$|\pi_0(X)| = \dim\{f \in C^1(U) \mid \mathrm{d}f = 0\}$$

because the vanishing of the derivation $\mathrm{d}f$ means that f is a locally constant function.

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Example (Counting pieces is NOT enough)

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Goal: find smooth functions that can detect holes (at least two-dimensional holes).

Observation from calculus: Green formula

Proposition

Let $L, l_1, l_2, \cdots, l_n$ be disjoint closed simple curves on $\mathbb{R}^2$ such that $l_1, \cdots, l_n$ are contained in the interior Ω_L of L. Let D be a subset of Ω_L such that $\partial D = L \coprod l_1 \coprod \cdots \coprod l_n$. Suppose $P(x, y)$ and $Q(x, y)$ are functions with continuous partial derivations, then

$$\iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \mathrm{d}x\mathrm{d}y = \oint_L + \oint_{l_1} + \cdots + \oint_{l_n} P\mathrm{d}x + Q\mathrm{d}dy$$

Smooth vector fields detect the shape of spaces

Definition

Let $U \subset \mathbb{R}^2$ be an open subset. A pair of smooth functions $f, g \colon U \to \mathbb{R}^2$ is called an **irrotational field**, if $\frac{\partial f}{\partial y} - \frac{\partial g}{\partial x} = 0$.
It is called **a potential field** if there exists a function $F \colon \mathbb{R}^2 \to \mathbb{R}$ such that $\frac{\partial F}{\partial x} = f$ and $\frac{\partial F}{\partial y} = g$.

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Proposition

The following assertions distinguish $\mathbb{R}^2$ and $\mathbb{R}^2 - 0$:

1. *Any irrotational field on $\mathbb{R}^2$ is a potential field.*
2. *There exists an irrotational field on $\mathbb{R}^2 - 0$ that is not a potential field. For example, $(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2})$.*

Reorganization of these observations

Note that the set of smooth vector fields (or irrotational fields, or potential fields) on U forms a vector space. We summarize the previous observation as

1. $\dim\{$irrotational fields on $\mathbb{R}^2\}/\{$potential fields on $\mathbb{R}^2\} = 0$.

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From this viewpoint, functions and vector fields will help us understand the shape of a space. Our goal is to develop this method systematically via differential forms.

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Differentials forms: 1-forms on vector spaces

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Example

Let $T_0\mathbb{R}^n = \left\langle \frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n} \right\rangle$ be a vector space. For any open neighbourhood $U \subset \mathbb{R}^n$ of 0 and any smooth function $f\colon U \to \mathbb{R}$, $(\mathrm{d}f)_0$ defines a 1-form

$$(\mathrm{d}f)_0\colon \frac{\partial}{\partial x_i} \mapsto \frac{\partial f}{\partial x_i}(0)$$

(Here $T_0\mathbb{R}^n$ is the tangent space of $\mathbb{R}^n$ at 0. Roughly speaking, the tangent space mean the space of derivations.)

Differentials forms: 1-forms on vector spaces

Definition

Given a vector space V, its dual space V^* is the space of 1-forms on V, namely $\mathrm{Hom}(V, \mathbb{R})$. Given a basis $(e_1, \cdots, e_n)$ of V, we define its dual basis $(\delta_1, \cdots, \delta_n)$ for V^* by setting

$$\delta_i(e_j) = \delta_{ij} \text{ (Kronecker delta)}$$

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Example

Let $x_i \colon \mathbb{R}^n \to \mathbb{R}$ be the projection $(a_1, \cdots, a_n) \mapsto a_i$. Then $\{\mathrm{d}x_i\}_{i=1}^n$ is the dual basis with respect to $\{\frac{\partial}{\partial x_i}\}_{i=1}^n$. From this viewpoint, we can understand why we write

$$\mathrm{d}f = \sum \frac{\partial f}{\partial x_i} \mathrm{d}x_i$$

The intuition of 1-forms

Example (gravitational work 1-form)

Fixed an object with mass, let $\mathbf{v} \in \mathbb{R}^2$ be a vector

$$\omega(\mathbf{v}) := \text{work done moving the mass along } \mathbf{v}$$

One can check that ω is indeed a 1-form.

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Example

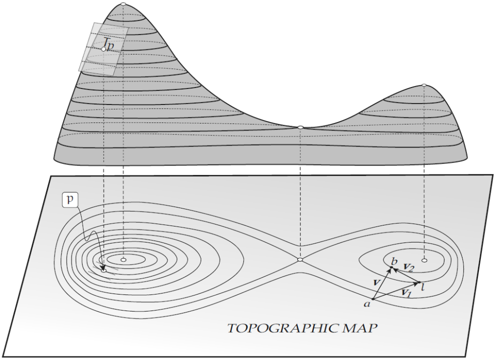
If we consider $\varphi \colon 2\mathrm{d}x + dy$ on $\mathbb{R}^2$, then the picture of this 1-form is given by the picture of isopotential lines with slope -0.5. One can imagine it as a picture of electric field intensity.

The intuition of 1-forms

Let $h\colon \mathbb{R}^2 \to \mathbb{R}$ be a smooth function. Let $a, b \in \mathbb{R}^2$ and $\mathbf{v} = \overrightarrow{ab}$, if we define

$$\eta(\mathbf{v}) := h(b) - h(a)$$

is η a 1-form?



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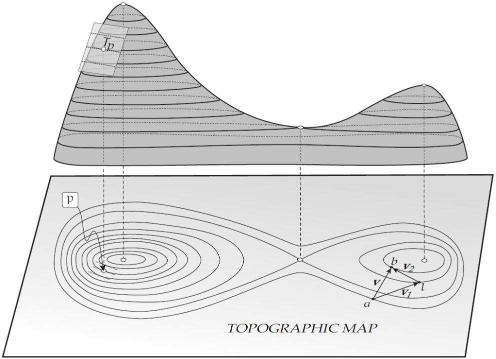
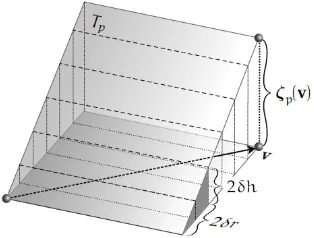
is η a 1-form? **NO!**



The intuition of 1-forms

The correct definition of the 1-form for h should be defined on each tangent plane. For $\mathbf{v} \in T_p$, $\zeta_p(\mathbf{v}) :=$ change of height along $\mathbf{v}$ on T_p.



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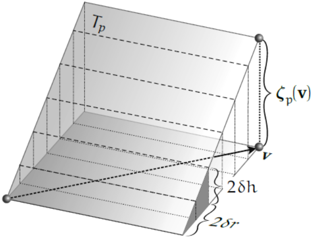
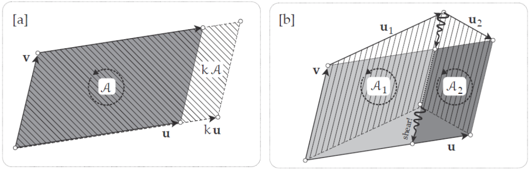


The field on T_p is given by $\mathrm{d}h_p = \frac{\partial h}{\partial x}(p)\mathrm{d}x + \frac{\partial h}{\partial y}(p)\mathrm{d}y$. From this viewpoint, we know why the gradient $(\frac{\partial h}{\partial x}(p), \frac{\partial h}{\partial y}(p))$ at p is the direction of most rapid increase of h.

The intuition of n-forms

Given $\mathbf{u}, \mathbf{v} \in \mathbb{R}^2$, we define

$$\mathcal{A}(\mathbf{u}, \mathbf{v}) = \text{oriented area of the parallelogram with edges } \mathbf{u} \text{ and } \mathbf{v}$$



It is a linear functional on $\mathbb{R}^2 \otimes \mathbb{R}^2$ such that $\mathcal{A}(\mathbf{u}, \mathbf{v}) = -\mathcal{A}(\mathbf{v}, \mathbf{u})$.

Similarly, oriented volume is an n-form on $\mathbb{R}^n$.

The definition of n-forms

Definition

Let V be a vector space. An n-form on V is a linear functional

$$\Psi\colon V^{\otimes n} \to \mathbb{R}$$

such that $\Psi(v_1, \cdots, v_n) = \operatorname{sgn}(\sigma)\Psi(v_{\sigma(1)}, \cdots, v_{\sigma(n)})$ for any $\sigma \in S_n$. The space of n-forms on V is denoted by $\mathrm{Form}^n(V)$.

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Proposition

Any n-form on $\mathbb{R}^n$ is a scalar of the determinant. In particular, $\dim \mathrm{Form}^n(\mathbb{R}^n) = 1$.

Tensor products and wedge products

Definition

For any two $\varphi, \psi \in V^*$, the **tensor product** $\varphi \otimes \psi \in (V^{\otimes 2})^*$ is defined to be

$$\varphi \otimes \psi(v \otimes u) = \varphi(v)\psi(u)$$

The **wedge product** $\varphi \wedge \psi$ is defined to be

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Remark

$$\begin{aligned}(\mathrm{d}x \wedge \mathrm{d}y)(\mathbf{u}, \mathbf{v}) =& (\mathrm{d}x \otimes \mathrm{d}y - \mathrm{d}y \otimes \mathrm{d}x)\left(\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}\right) \\ =& u_1 v_2 - u_2 v_1 = \det \begin{bmatrix} u_1 & v_1 \\ u_2 & v_2 \end{bmatrix} \\ =& \mathcal{A}(\mathbf{u}, \mathbf{v})\end{aligned}$$

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Definition

The wedge product $\wedge : \mathrm{Form}^p(V) \times \mathrm{Form}^q(V) \to \mathrm{Form}^{p+q}(V)$ is defined to by

$$(\omega_1 \wedge \omega_2)(v_1, \cdots, v_{p+q}) = \sum_{\sigma \in S(p,q)} \mathrm{sgn}(\sigma)\omega_1(v_{\sigma(1)}, \cdots, v_{\sigma(p)})\omega_2(v_{\sigma(p+1)}, \cdots, v$$

where $S(p, q) \subset S_{p+q}$ is the subset of (p, q) shuffle of $\{1, \cdots, p+q\}$ i.e.

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Remark

In this way, $\mathrm{Form}^*(V) = \bigoplus_n \mathrm{Form}^n(V)$ *is an anti-commutative graded* $\mathbb{R}$*-algebra.*

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Differential forms on spaces

Definition

Let $U \subset \mathbb{R}^n$ be an open subset. A differential k-form ω on U is a smooth map

$$\omega \colon U \to \mathrm{Form}^k(\mathbb{R}^n)$$

Note that a 0-form is a smooth function on U. The space of differential k-forms on U is denoted by $\Omega^k(U)$.

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Remark

We may write a differential k-form ω by

$$\omega = \sum f_I \mathrm{d}x_I$$

where I is an ordered set $\{i_1 < \cdots < i_k\}$ of $\{1, \cdots, n\}$ and $\mathrm{d}x_I$ means $\mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_k}$.

Operations on differential forms

Definition

Suppose $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^n$ are two open subsets, for any smooth map $f\colon U \to V$, the pull-back $f^* : \Omega^k(V) \to \Omega^k(U)$ is defined by pre-composed

$$f^* : \left(V \xrightarrow{\omega} \mathrm{Form}^k(\mathbb{R}^n) \right) \mapsto \left(U \xrightarrow{f} V \xrightarrow{\omega} \mathrm{Form}^k(\mathbb{R}^n) \right)$$

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Definition (differential operator)

Suppose $\omega = \sum f_I \mathrm{d}x_I \in \Omega^k(U)$, the differential operator $\mathrm{d} \colon \Omega^k(U) \to \Omega^{k+1}(U)$ is defined by

$$\mathrm{d} \colon \sum f_I \mathrm{d}x_I \mapsto \sum \mathrm{d}f_I \wedge \mathrm{d}x_I$$

De Rham cohomology

Lemma

The composition $\Omega^{k-1}(U) \xrightarrow{d} \Omega^{k}(U) \xrightarrow{d} \Omega^{k+1}(U)$ *is the zero map.*

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Definition (de Rham cohomology theory)

Suppose $U \subset \mathbb{R}^n$, the **de Rham complex** $\Omega^*(U)$ is

$$0 \to \Omega^0(U) \xrightarrow{d} \Omega^1(U) \xrightarrow{d} \Omega^2(U) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n(U)$$

The q-th **de Rham cohomology** $H^q_{DR}(U)$ of U is $H^q\Omega^*(U)$.
A differential k-form ω is **closed** if $\mathrm{d}\omega = 0$; ω is **exact** if $\omega = \mathrm{d}\psi$ for a differential $k-1$-form ψ.

Example: de Rham cohomology for $\mathbb{R}^2$ and $\mathbb{R}^2 - 0$

Example

$$H^i_{DR}(\mathbb{R}^2) = \begin{cases} \mathbb{R}, & i = 0 \\ 0, & \text{otherwise} \end{cases}$$

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Example

Now we show $H^1_{DR}(\mathbb{R}^2 - 0) = \mathbb{R}$: let S^1 be the unit circle in $\mathbb{R}$, define

$$\int_{S^1} : H^1_{DR}(\mathbb{R}^2 - 0) \to \mathbb{R}$$

We just need to show it is injective. Suppose ω is a closed 1-form such that $\int_{S^1} \omega = 0$. We claim that for each closed curve C in $\mathbb{R}^2 - 0$, $\int_C \omega = 0$. Then ω will be a conservative field and thus exact.

Mayer-Vietoris property

Proposition

Suppose $U, V \subset \mathbb{R}^n$ *and let* $i\colon U \hookrightarrow U \cup V$ *and* $j\colon V \hookrightarrow U \cup V$. *Then there is a short exact sequence for de Rham complexes*

$$0 \to \Omega^*(U \cup V) \xrightarrow{(i^*, j^*)} \Omega^*(U) \oplus \Omega^*(V) \xrightarrow{\psi} \Omega^*(U \cap V) \to 0$$

where $\psi(\omega, \tau) = \tau - \omega$.

This will induce a long exact sequence for de Rham cohomology groups:

$$\cdots \to H^i_{DR}(U) \oplus H^i_{DR}(V) \to H^i_{DR}(U \cap V) \to H^{i+1}(U \cup V) \to \cdots$$

The notion of homotopy

Definition

Let $f, g\colon X \to Y$ be two continuous maps. We say f is homotopic to g, if there exists a continuous map $H\colon X \times I \to Y$ such that $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$. We denote it by $f \sim_h g$.
Suppose f, g are smooth, we say f is smooth homotopic to g if H is also smooth.

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Definition

$f\colon X \to Y$ is a homotopy equivalence if there exists a continuous map $g\colon Y \to X$ such that $f \circ g \sim_h \mathrm{id}_Y$ and $g \circ f \sim_h \mathrm{id}_X$. We say X is homotopy equivalent to Y if there exists a homotopy equivalence $X \to Y$.

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Example

$\mathbb{R}^n - 0$ is homotopy equivalent to S^{n-1}.

Homotopy invariant property

Proposition

Let $p\colon \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ *be the projection. Then induced pull-back* $p^*: H^*_{DR}(\mathbb{R}^n) \to H^*_{DR}(\mathbb{R}^n \times \mathbb{R})$ *is an isomorphism.*

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Corollary (Poincare lemma)

$H^i_{DR}(\mathbb{R}^n) = 0$ *for* $i > 0$. *In other words, any closed form on* $\mathbb{R}^n$ *is exact.*

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Corollary

If U *and* V *are homotopy equivalent, then* $H^*_{DR}(U) \simeq H^*_{DR}(V)$.

de Rham cohomology of spheres

Proposition

Let S^n be an n-dimensional sphere. Then $H^i_{DR}(S^n) = \mathbb{R}$ if and only if $i = 0, n$.

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Proof.

We sketch the proof in the following way:

1. Use the homotopy invariant property to show that $H^1_{DR}(S^1) \cong H^1_{DR}(\mathbb{R}^2 - 0) \cong \mathbb{R}$.

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4. Use the long exact sequence by Mayer-Vietoris property and this open cover.

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de Rham cohomology of linear isomorphisms

Since $\mathbb{R}^n - 0 \simeq S^{n-1}$, we have $H^*_{DR}(\mathbb{R}^n - 0) = \mathbb{R}$. Let A be an $n \times n$ invertible matrix and define $f_A : \mathbb{R}^n - 0 \to \mathbb{R}^n - 0$ by $\mathbf{v} \mapsto A\mathbf{v}$.

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For each $n \geq 2$, the induced map $f_A^* : H^{n-1}_{DR}(\mathbb{R}^n - 0) \to H^{n-1}_{DR}(\mathbb{R}^n - 0)$ is a multiplication by $\det A / |\det A|$.

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4. Use Mayer-Vietoris property.

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Application: vector fields on spheres

Theorem

The sphere S^n has a tangent vector field v with $v(x) \neq 0$ for $x \in S^n$ if and only if n is odd.

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Proof.

Suppose such a tangent vector field v exists, we may extend it to a map $f\colon \mathbb{R}^{n+1} - 0 \to \mathbb{R}^{n+1} - 0$ by $x \mapsto v(x/||x||)$ (here we may embed S^n into $\mathbb{R}^{n+1} - 0$). Note that x and $v(x)$ are orthogonal. Then we have $F(x, t) = (\cos \pi t)x + (\sin \pi t)v(x)$ that defines a homotopy from id_{S^n} to $f_{\mathrm{diag}(-1,\cdots,-1)}$. By previous calculation, $f^*_{\mathrm{diag}(-1,\cdots,-1)}$ is a multiplication by $(-1)^{n+1}$, which forces that n must be odd.
Conversely, for $n = 2m - 1$, consider

$$v(x_1, x_2, \cdots, x_{2m}) = (-x_2, x_1, -x_4, x_3, \cdots, -x_{2m-1}, x_{2m-1})$$

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Just kidding! Adams solved this problem completely in 1962 using K-theory and cohomology operations on K-theory (so-called Adams operations).

Outline

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3. de Rham Cohomology Theory and its Applications
4. Summary

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- The generalization of functions on a space is the notion of sheaves on a space.
- Differential forms and de Rham cohomology can be defined on any differentiable manifolds, even algebraic varieties.
- Roughly speaking, a cohomology theory assigns each space a graded algebra satisfying the Mayer-Vietoris property and homotopy property.

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