An Introduction to Algebraic Geometry

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2022.12.16

Basic setup

Definition:
Affine n-space $\mathbb{A}^n := \{(a_1, a_2, ..., a_n) : a_i \in K\}$
Consider subset $S \subset K[x_1, ..., x_n]$ and the zero locus of S
$V(S) := \{x \in \mathbb{A}^n : f(x) = 0, \forall f \in S\}$
If S finite , say $S = \{f_1, ..., f_k\}$ We write $V(S) = V(f_1, ..., f_k)$

Basic setup

Example:

(a) $V(0) = \mathbb{A}^n$

(b) $V(1) = \emptyset$

(c) let $a = (a_1, ..., a_n)$, $V(x_1 - a_n, ..., x_n - a_n) = \{a\}$

Interaction with Set Operation

Proposition:

(a) $S_1 \subset S_2 \Rightarrow V(S_1) \supset V(S_2)$

(b) $V(S_1) \bigcup V(S_2) = V(S_1 S_2)$

(c) $\bigcap_{i \in \mathcal{J}} V(S_i) = V(\bigcup_{i \in \mathcal{J}} S_i)$

More precise expression

Observation: $V(S)=V(<S>)$

Indeed, for $\forall f, g \in S$ and $\forall h \in K[x_1, ..., x_n]$

We always have $(f+g)(x)=0$ and $h \cdot f\ (x)=0\ \forall x \in V(S)$

Thus we may view varieties as loci of ideals

More precise expression

Proposition:

(a) $V(\mathcal{J}) = V(\sqrt{\mathcal{J}})$

(b) $V(\mathcal{J}_1) \bigcup V(\mathcal{J}_2) = V(\mathcal{J}_1\mathcal{J}_2) = V(\mathcal{J}_1 \bigcap \mathcal{J}_2)$

(c) $V(\mathcal{J}_1) \bigcap V(\mathcal{J}_2) = V(\mathcal{J}_1 \bigcup \mathcal{J}_2) = V(\mathcal{J}_1 + \mathcal{J}_2)$

This relates geometric objects to an algebraic objects

Literally assigns an ideal to a variety

The converse assignment

Definition:
The ideal of X is $I(X) := \{f \in K[x_1, ..., x_n] : f(x) = 0, \forall x \in X\}$
Remark: $I(X)$ is radical

The main theorem

Hilbert's Nullstallensatz Theorem:

$\{affine\ varieties\ in\ \mathbb{A}^n\} \stackrel{1-1}{\longleftrightarrow} \{radical\ ideals\ in\ K[x_1, ..., x_n]\}$

some examples:...

Analogous properties of I(·)

Proposition:

(a) $I(X_1 \bigcup X_2) = I(X_1) \bigcap I(X_2)$

(b) $I(X_1 \bigcap X_2) = \sqrt{I(X_1) + I(X_2)}$

Example:

For $X_1 = V(x_1^2 - x_2)\ X_2 = V(x_2)$ consider $I(X_1 \bigcap X_2)$ and $X_1 \bigcap X_2$

Few Remarks

a) Weak Nullstellensatz: for proper idea $\mathcal{J}$, $\mathcal{J}$ has zero
Otherwise:
$\sqrt{\mathcal{J}} = I(V(\mathcal{J})) = I(\emptyset) = K[x_1, ..., x_n] = (1) \Rightarrow \mathcal{J} = (1)$

b) Polynomial and function on $\mathbb{A}^n$ agrees,since
$f - g \in I(\mathbb{A}^n) = \sqrt{(0)} = (0)$
This motivates us to consider the functions defined on a certain variety X

Substructure

Definition: polynomial function on X is a map $X \to K$ that is of the form $x \mapsto f(x)$ for some $f \in K[X_1, ..., Xn]$

Indeed the ring of all polynomial function on X is just $A(X) := K[x_1, ..., x_n]/I(X)$ called coordinate ring of X

a) For $S \in A(Y), V_Y(S) := \{x \in Y : f(x) = 0, \forall f \in S\}$ called affine subvarieties of Y

b) $I_Y(X) := \{f \in A(Y) : f(x) = 0, \forall x \in X\}$

1. Hillbert Nullstellensatz
2. Zariski Topology
3. Regualr function

Basic setup

Definition: For affine variety X , let closed sets in X be affine subvarieties of X(Axioms check?)

Important geometric objects

Definition:
X reducible if $X = X_1 \bigcup X_2$ where X_1, X_2 proper and closed in X
Otherwise, call X irreducible
X disconnected if $X = X_1 \bigcup X_2$ where X_1, X_2 proper closed and disjoint in X Otherwise, call X connected

Important geometric objects

Remark:for disconnected $X = X_1 \bigcup X_2 \quad A(X) \cong A(X_1) \times A(X_2)$
Note $\sqrt{I(X_1) + I(X_2)} = I(X_1 \bigcap X_2) = (1)$ i.e. $I(X_1) + I(X_2) = (1)$
$I(X_1) \bigcap I(X_2) = I(X_1 \bigcup X_2) = I(X) = (0)$
Then Chinese Remainder Theorem finishes the proof

Important geometric objects

Proposition: X reducible $\Leftrightarrow$ there's zero divisor in $A(X)$
Remark:X irreducible $\Leftrightarrow A(X) = A(Y)/I(X)$ is integral domain
In other word:
$\{irreducible\ subvarieties\ in\ Y\} \xleftrightarrow{1-1} \{prime\ ideals\ in\ A(Y)\}$

Irreducible decomposition of variety

One may wonder if an arbitrary variety can be represented as a union of irreducible subvarieties

However, this requires suitable finiteness condition

Definition: topological space X is Noetherian if any nested closed sequence $X_0 \supset X_1 \supset X_2 \supset ...$ will stationary

Irreducible decomposition of variety

Observation: affine variety is Neotherian

Irreducible decomposition of variety

Theorem:

Irreducible decomposition of variety ,say $X = \bigcup_{i=0}^{r} X_i$,exists and is unique if $X_i \nsubseteq X_j, \forall i, j$

Irreducible decomposition of variety

Remark:Primary decomposition gives an irreducible decomposition
$I(X) = Q_1 \bigcap ... \bigcap Q_r$
then X=V(I(X))=V(Q_1)$\bigcup ... \bigcup V(Q_r) = V(P_1) \bigcup ... \bigcup V(P_r)$
where $P_i = \sqrt{Q_i}$ prime
In other word:
$\{irreducible\ componentofX\} \stackrel{1-1}{\longleftrightarrow} \{minimal\ prime\ ideals\ in\ A(X)\}$

Open set in irreducible space

Striking fact: open sets are dense in irreducible space
To some extend, open set tends to be very big in Zariski topology
Indeed no further decomposition indicates that the intersection of any two open sets is nonempty!

Introduction

To investigate the morphism between varieties, analogous to manifold, we adopt a local and media approach. This leads us to the study of correct function on varieties

locally defined function

Definition: for U open in Y, map $\varphi : U \to K$ is regular if:
$locally \forall a \in U\, there's\, a\, neighborhood\, U_a\, with\, \varphi = \frac{g}{f}\, on\, U_a$
where f(x)=0 on $U_a, f, g \in A(X)$
Denote all regular functions on U as $\mathcal{O}_X(U)$

Identity Theorem

Lemma : $V(\varphi)$ is closed in U
Corollary : φ_1, φ_2 coincide on open $U \Rightarrow$ they coincides on $\overline{U}$
Remark : analogous to holomorphic function in complex analysis where open sets are small

Regular function on basic brick

Definition : distinguished open set $D(f) = X - V(f), f \in A(X)$
Observation : distinguished open set behaves nicely w.r.t. union and intersection:

a) $D(f) \bigcap D(g) = D(fg)$

b) $U = X - V(f_1, ..., f_k) = X - \bigcap_{i=1}^{k} V(f_i) = \bigcup_{i=1}^{k} D(fi)$

Indeed, those are the behaviors of basis!

Regular function on basic brick

Theorem : $\mathcal{O}_X(D(f)) = \left\{\frac{g}{f^n} : g \in A(x), n \in \mathbb{N}\right\}$
This implies regular functions behaves uniformly on each "micro-component"
Corollary : regular function as localization $\mathcal{O}_X(D(f)) \cong A(X)_{(f)}$

Regular function on basic brick

Example: $\mathcal{O}_{\mathbb{A}^2}(\mathbb{A}^2 - 0) = K[x_1, x_2]$ thus $\mathcal{O}_X(U) = \mathcal{O}_X(X)$
Remark: in fact gives a extension which is analogous to Removable Singularity Theorem in complex analysis

More sophisticated view for regular function

Definition: A presheaf F on topology X consists two data:
equips open set U with a ring $\mathcal{F}(U)$
brings inclusion $U \subset V$ with map on ring equipped
$\rho_{U,V} : \mathcal{F}(V) \to \mathcal{F}(U)$ as restriction map
satisfying $\{(\emptyset) = 0 \quad \rho_{U,V} = id_{F(U)} \quad associativity$
example: $\mathcal{O}_X$ is the sheaf of regular functions on X

Thanks!