Def affine space $\mathbb{A}^n := \{(a_1, \dots, a_n) : a_i \in K\}$

subset $S \subseteq K[x_1, \dots, x_n]$

zero locus of $S$ $V(S) := \{x \in \mathbb{A}^n : f(x) = 0, \forall f \in S\}$

subsets of $\mathbb{A}^n$ of this form are called varieties

if $S$ finite say $S = \{f_1, \dots, f_k\}$ write $V(S) = V(f_1, \dots, f_k)$

interaction with set operation : (a) $S_1 \subseteq S_2$ $V(S_1) \supseteq V(S_2)$

(b) $V(S_1) \sqcup V(S_2) = V(S_1 S_2)$ (c) $\bigsqcap_{i \in J} V(S_i) = V(\bigsqcup_{i \in J} S_i)$

View $S$ as requirement

e.g. $\mathbb{A}^n = V(0)$ $\phi = V(1)$ $a = (a_1, \dots, a_n)$ $\{a\} = V(x_1 - a_1, \dots, x_n - a_n)$

Obs. $V(S) = V(\langle S \rangle)$ for $f, g \in V(S)$ $f + g \in V(S)$ $r \cdot g \in V(S)$

View varieties as loci of ideals

$$S \underset{\text{Noetherian}}{\overset{\text{zero element}}{\rightleftarrows}} \langle S \rangle$$

To find a more exact expression for statement above:

(a) $V(J) = V(\sqrt{J})$

(b) $V(J_1) \sqcup V(J_2) = V(J_1 \sqcap J_2)$

(c) $V(J_1) \sqcap V(J_2) = V(J_1 + J_2)$

This relates geometric objects to algebraic objects

Literally assigns a ideal to a variety What about the converse

Def. ideal of $X$ $I(X) := \{f \in K[x_1, \dots, x_n] : f(x) = 0 \text{ for all } x \in X\}$

Remark: Once again requirement and $I(X)$ radical

Now consider what $I(\cdot)$ and $V(\cdot)$ can do

Thm. Hilbert's Nullstellensatz $\overline{K} \overset{(b)}{=} K$

$\{\text{affine varieties in } \mathbb{A}^n\} \overset{1-1}{\longleftrightarrow} \{\overset{(a)}{\text{radical}} \text{ ideals in } K[x_1, \cdots, x_n]\}$

Proof: To show $V(I(X)) \supseteq X$

$x \in X$ then $f(x) = 0$ for all $f \in I(X)$ i.e. $x \in V(I(X))$

To show $I(V(J)) \supseteq \sqrt{J}$

$f \in \sqrt{J}$ $f^k \in J$ $f^{(k)}(x) = 0$ for all $x \in V(J)$

then $f(x) = 0$ for all $x \in V(J)$ thus $f \in I(V(J))$

To show $V(I(X)) \subseteq X$

say $X = V(J)$ $I(V(J)) \supseteq \sqrt{J} \supseteq J \xrightarrow{V(\cdot)} V(I(X)) \subseteq V(J) = X$

To show $I(V(J)) \subseteq \sqrt{J}$ which is hard $\square$

e.g. (a) in $\mathbb{A}^1$ PID $K[x_1]$ $J = \langle f \rangle$ say $f = (x_1 - a_1)^{k_1} \cdots (x_1 - a_r)^{k_r}$

$V(J) = V(f) = \{a_1, \cdots, a_r\}$ last the date of multiplicities

in other word $I(V(J)) = \sqrt{J} = \langle (x_1 - a_1) \cdots (x_1 - a_r) \rangle$

(b) prime hence radical $J = \langle x_1^2 + 1 \rangle$ in $\mathbb{R}[x_1]$ $\overline{\mathbb{R}} = \mathbb{C}$

but $I(V(J)) = I(\phi) = \mathbb{R}[x_1] \neq J = \sqrt{J}$ where Thm. breaks down

(c) $J = \langle x_1 - a_1, \cdots, x_n - a_n \rangle$ maximal hence radical in $K[x_1, \cdots, x_n]$

$V(J) = \{a\}$ $I(\{a\}) = I(V(J)) = J = \langle x_1 - a_1, \cdots, x_n - a_n \rangle$

$\{\text{points in } \mathbb{A}^n\} \overset{1-1}{\longleftrightarrow} \{\text{maximal ideals in } K[x_1, \cdots, x_n]\}$

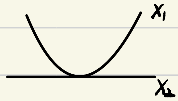
This perspective translate properties of $V(\cdot)$ into $I(\cdot)$'s

Prop. (a) $I(X_1 \sqcup X_2) = I(X_1) \sqcap I(X_2)$ by def

(b) $I(X_1 \sqcap X_2) = \sqrt{I(X_1) + I(X_2)}$

Proof $I(X_1 \sqcap X_2) = I(V(I(X_1)) \sqcap V(I(X_2))) = I(V(I(X_1) + I(X_2))) = \sqrt{I(X_1) + I(X_2)}$

e.g. find radical of $\langle x_1^3 - x_2^6, x_1 x_2 - x_2^3 \rangle$




consider $X_1 = V(x_2^2 - x_1)$ $X_2 = V(x_2)$

$I(X_1 \sqcap X_2) = \sqrt{\langle x_2^2 - x_1, x_2 \rangle} = \langle x_1, x_2 \rangle$ $X_1 \sqcap X_2 = V(x_1, x_2) = 0$

Remark a) Weak Nullstellensatz : for proper $J$ $J$ has zero

Otherwise $\sqrt{J} = I(V(J)) = I(\phi) = K[x_1, \dots, x_n]$ $\sqrt{J} = (1)$ implies $J = (1)$

b) polynomial and function on $\mathbb{A}^n$ agrees

$f - g \in I(\mathbb{A}^n) = \sqrt{\langle 0 \rangle} = \langle 0 \rangle$

$\downarrow$ in case of $X$

Substructure

Def polynomial function on $X$ is a map $X \to K$ that is of the form $x \mapsto f(x)$ for some $f \in K[x_1, \dots, x_n]$

Indeed the ring of all polynomial function on $X$ is just $A(X) := K[x_1, \dots, x_n]/I(X)$ call it coordinate ring

(a) For $S \subseteq A(Y)$ $V_Y(S) := \{x \in Y : f(x) = 0 \text{ for all } f \in S\}$

affine subvarieties of $Y$

(b) $I_Y(X) := \{f \in A(Y) : f(x) = 0 \text{ for all } x \in X\}$

Zariski topology

Def. for affined variety $X$, let closed sets in $X$ be affine subvarieties of $X$

Def $X$ reducible if $X = X_1 \sqcup X_2$ proper closed
otherwise irreducible

$X$ disconnected if $X = X_1 \sqcup X_2$ proper closed disjoint
otherwise connected

Remark disconnected $X = X_1 \sqcup X_2$ $A(X) \cong A(X_1) \times A(X_2)$

Notice $\sqrt{I(X_1) + I(X_2)} = I(X_1 \sqcap X_2) = (1)$ i.e. $I(X_1) + I(X_2) = (1)$
$I(X_1) \sqcap I(X_2) = I(X_1 \sqcup X_2) = I(X) = (0)$ then by C.R. Thm.

Prop. $X$ reducible $\Leftrightarrow$ $\exists$ zero-divisor in $A(X)$

$\Rightarrow$ for $X = X_1 \sqcup X_2$ $I(X_i) \neq (0)$ choose $0 \neq f_i \in I(X_i)$
then $f_1 f_2$ vanishes on $X_1 \sqcup X_2 = X$ Hence $f_1 f_2 = 0$

$\Leftarrow$ Let non-zero $f_1, f_2 \in A(X)$ and $f_1 f_2 = 0$ then $X = V(0)$
$= V(f_1 f_2) = V(f_1) \sqcup V(f_2)$ reducible

Remark irr. $X$ in $Y$ $\Leftrightarrow$ $A(X) = A(Y)/I(X)$ integral domain

i.e. $\{$irr. subvarieties$\} \leftrightarrow \{$primes ideals in $A(Y)\}$

One may wonder an arbitrary variety can be represented this requires suitable finiteness property :

Def topological space $X$ Noetherian if nested closed sequence $X_0 \supseteq X_1 \supseteq \cdots$ stationary

Obs. Affine variety is Noetherian

$X_0 \supseteq X_1 \supseteq \cdots$ corresponds to $I(X_0) \subseteq I(X_1) \cdots$

the latter stationary since $A(X)$ Noetherian

Irr. decomposition $\exists!$ $X = \bigsqcup_{i=1}^{r} X_i$ if $X_i \not\subset X_j$

$\exists$: Otherwise pair decomposition produces an infinite chain (x)

$!$: $X_1 \sqcup \cdots \sqcup X_r = X_1' \sqcup \cdots \sqcup X_s'$

irr. $X_i = \bigsqcup_j (X_i \sqcap X_j')$ forces $X_i = X_i \sqcap X_j'$ i.e. $X_i \sqsubset X_j'$

Similarly $X_j' \sqsubset X_k$ then $X_i \sqsubset X_k$ i.e. $X_i = X_j'$

Primary decomposition gives an irr. decomposition

$I(X) = Q_1 \sqcap \cdots \sqcap Q_r$ then $X = V(I(X)) = V(Q_1) \sqcup \cdots \sqcup V(Q_r)$

$= V(P_1) \sqcup \cdots \sqcup V(P_r)$ where $P_i = \sqrt{Q_i}$ prime

$\{$irr. components$\} \leftrightarrow \{$ minimal prime ideal in $A(X)\}$

Open sets are dense in irr. space

open set tends to be very big in Zariski topology

Indeed no further decomposition indicates open intersection nonempty

To investigate the morphism between varieties, analogous to manifold, we adapt a local and media approach. but componentwisely. Thus we may study suitable function on variety

Def for $U$ open in $X$ map $\varphi: U \to K$ regular if: locally $\forall a$ there's a neighborhood $U_a$ with $\varphi = \frac{g}{f}$ on $U_a$ where $f(x) \neq 0$ on $U_a$ $f, g \in A(X)$ Locally defined

denote all regular functions on $U$ as $\mathcal{O}_X(U)$

e.g. $X = V(x_1x_4 - x_2x_3)$ $U = X \setminus V(x_2, x_4)$

Let $\varphi: U \to K$, $(x_1, x_2, x_3, x_4) \mapsto \begin{cases} \frac{x_1}{x_2} & \text{if } x_2 \neq 0 \quad U_1 \\ \frac{x_3}{x_4} & \text{if } x_4 \neq 0 \quad U_2 \end{cases}$

compatible in $U_1 \cap U_2$ since in $X$

To find out whether two regular functions are same or not, we may study $V(\varphi_1 - \varphi_2)$ i.e. the zero locus of a regular function

Coincide with intuition: patching $V(g_a)$ up as $V(\varphi)$

Lem. $V(\varphi)$ closed in $U$

$U_a \setminus V(\varphi) = \{x \in U_a : \varphi(x) \neq 0\} = U_a \setminus V(g_a)$ open

patching $\bigsqcup_{a \in U} U_a \setminus V(\varphi) = U \setminus V(\varphi)$ open

Col Identity Thm. for regular functions

$\varphi_1, \varphi_2$ coincide on $U$ open $\Rightarrow$ so as on $\bar{U}$

$U \subset V(\varphi_1 - \varphi_2)$ closed $\Rightarrow$ $\bar{U} \subset V(\varphi_1 - \varphi_2)$

Remark: analogous to holomorphic function in complex analysis where open sets are small

To investigate $\mathcal{O}_X(U)$ further we may study more fundamental bricks of open sets first:

Def distinguished open set $D(f) = X \setminus V(f)$ $f \in A(X)$

Obs distinguished open set behaves nicely w.r.t. union and intersection : (a) $D(f) \sqcap D(g) = D(fg)$ basis property

(b) $U = X \setminus V(f_1, \dots, f_k) = X \setminus \prod_{i=1}^{k} V(f_i) = \bigsqcup_{i=1}^{k} D(f_i)$

behaves uniformly on

Prop. $\mathcal{O}_X(D(f)) = \{ \frac{g}{f^n} : g \in A(X), n \in \mathbb{N} \}$ each micro-portions

$\supseteq$: To see what's more in this case

Say $a \in U_a$ $\varphi = \frac{g_a}{f_a}$ on $U_a$ then $a \in D(h_a) \subseteq U_a$ for some $h_a \in A(X)$ in particular $\varphi = \frac{g_a}{f_a} = \frac{g_a h_a}{f_a h_a}$ on $D(h_a)$ replace $f_a$ by $f_a h_a$ then

In other word, both numerator and denominator of $\varphi$ vanish on $V(h_a)$ but the latter doesn't on $D(h_a)$ i.e. $V(h_a) = V(f_a)$ $h_a = f_a$ in $A(X)$

by refining construction above $g_a f_b = g_b f_a$ $\forall a, b \in D(f)$

consider $\varphi = \frac{g_a}{f_a} = \frac{g_b}{f_b}$ on $D(f_a) \sqcap D(f_b)$

$g_a = f_a = 0$ on $V(f_a)$ $g_b = f_b = 0$ on $V(f_b)$

On the other hand $D(f) = \bigsqcup_{a \in D(f)} D(f_a)$ i.e.

$V(f) = \sqcap V(f_a) = V((f_a)_{a\in D(f)})$ then $f \in I(V(f)) = \sqrt{(f_a)_{a\in D(f)}}$

thus $f^n = \sum_a k_a f_a$ let $g = \sum_a k_a g_a$

uniform behavior : $\forall\, b \in D(f)$ $\varphi = \frac{g_b}{f_b} = \frac{g}{f^n}$

Indeed $f^n g_b = \sum_a k_a f_a g_b = \sum_a k_a g_a f_b = g f_b$

Remark: In particular, $X = D(1)$ then $\mathcal{O}_X(X) = A(X)$

Col regular function as localization $\mathcal{O}_X(D(f)) \cong A(X)_{(f)}$

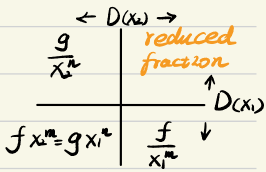
K-algebra homomorphism $A(X)_f \to \mathcal{O}_X(D(f))$, $\frac{g}{f^n} \mapsto \frac{g}{f^n}$

well-defined: $\frac{g}{f^n} = \frac{g'}{f^m}$ in $A(X)_f$ $f^k(gf^m - g'f^n) = 0$

on $D(f)$ this implies $gf^m = g'f^n$ i.e. $\frac{g}{f^n} = \frac{g'}{f^m}$ on $D(f)$

surjectivity by above prop. injectivity: for $\frac{g}{f^n} = 0$
on $D(f)$ then $g = 0$ on $D(f)$ then $fg = 0$ all over $X$
gives $f(g\cdot 1 - 0\cdot f^n) = 0$ i.e. $\frac{g}{f^n} = \frac{0}{1}$ in $A(X)_f$

e.g. $\mathcal{O}_{\mathbb{A}^2}(\mathbb{A}^2 \setminus \{0\}) = K[x_1, x_2]$ thus $\mathcal{O}_X(U) = \mathcal{O}_X(X)$
as an extension analogues to Removable Singularity Thm. in complex analysis ($f: \mathbb{C}\setminus\{0\} \to \mathbb{C}$ to $\tilde{f}: \mathbb{C} \to \mathbb{C}$)




$D(x_1) \sqcap D(x_2) \sqsubset V(f x_2^m - g x_1^n)$ thus
$\overline{D(x_1) \sqcap D(x_2)} = \mathbb{A}^2 \sqsubset V(f x_2^m - g x_1^n)$
i.e. $f x_2^m = g x_1^n$ on $A(\mathbb{A}^2) = K[x_1, x_2]$ UFD
this forces $m = n = 0$ $f = g$ i.e. $\varphi$ is a polynomial

Def presheaf $\mathcal{F}$ on topology $X$
- equips open set $U$ a ring $\mathcal{F}(U)$
- brings inclusion $U \sqsubseteq V$ with map on ring

equipped $\rho_{V,U} : \mathcal{F}(V) \to \mathcal{F}(U)$ as restriction

s.t. $\mathcal{F}(\phi) = 0$ $\rho_{U,U} = id_{\mathcal{F}(U)}$ associative on restriction

moreover $\mathcal{F}$ is a sheaf if satisfying gluing property:
$\forall$ open cover $\{U_i : i \in I\}$ compatible sections $\varphi_i \in \mathcal{F}(U_i)$
$\exists! \ \varphi \in \mathcal{F}(U)$ s.t. $\varphi|_{U_i} = \varphi_i$

e.g. $\mathcal{O}_X$ the sheaf of regular functions on $X$

Def restriction of $\mathcal{F}$ to $U$

Construction: Stalk of presheaf at $a$

$\mathcal{F}_a : \{ (U, \varphi) : U$ neighborhood of $a$ and $\varphi \in \mathcal{F}(U) \} / \sim$

$(U, \varphi) \sim (U', \varphi')$ if $a \in V \sqsubseteq U \sqcap U'$ $\varphi|_V = \varphi'|_V$

elements of $\mathcal{F}_a$ are germs (image them as regular function defines in an arbitrary small neighborhood of $a$

Lem. $\mathcal{O}_{X,a} \cong \{ \frac{g}{f} : f(a) \neq 0 \}$ local with maximal ideal

$I_a := \{ \overline{(U, \varphi)} \in \mathcal{O}_{X,a} : \varphi(a) = 0 \} \cong \{ \frac{g}{f} : g(a) = 0$ and $f(a) \neq 0 \}$

Proof: $A(X)_{I_a} \to \mathcal{O}_{X,a}$ ; $\frac{g}{f} \mapsto \overline{(D(f), \frac{g}{f})}$