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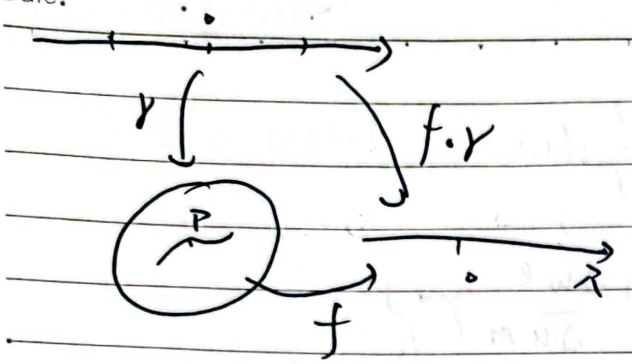


$\gamma \in \Gamma_p$

$\mathcal{F}_p = C_p^\infty/\sim \; = \{[f] \mid f \in C_p^\infty\}$

$\langle\langle \gamma, [f] \rangle\rangle = \frac{d}{dt}(f\circ\gamma)\big|_{t=0}$

$\mathcal{H}_p = \{[f] \in \mathcal{F}_p \mid \langle\langle \gamma, [f] \rangle\rangle = 0, \ \forall \gamma \in \Gamma_p\}$

$\mathcal{F}_p / \mathcal{H}_p = T_p^*(M)$

$(df)_p$

$= f(u^1 \cdots u^m)$

$(df)_p = \left(\frac{\partial f}{\partial u^i}\right)_p \cdot (du^i)_p$

$\dim T_p^*(M) = m$

$\gamma \quad \gamma' \in \Gamma_p \qquad \gamma \sim \gamma'$

for any $(df)_p \in T_p^*(M)$

$\langle\langle \gamma, \ \rangle\rangle = \langle\langle \gamma', \ \rangle\rangle$

$\Gamma_p/\sim \; = T_p(M)$

~~with~~ $u^i(t) = u^i(0) + \gamma^i t. \qquad \gamma^i = \frac{du^i}{dt}$

$\langle [\lambda_k], (df)_p \rangle = \frac{\partial f}{\partial u^k}$

$\dim T_p(M) = m$

denote $[\lambda_k] = \frac{\partial}{\partial u^k}$ basis of $T_p(M)$ $[\gamma] = \sum \gamma^i \frac{\partial}{\partial u^i}$

differential forms:

tensor product:

$$\begin{tikzcd} V\times W \arrow[r, "h"] \arrow[dr, "f"'] & Y \arrow[d, "g\ \exists!"] \\ & Z \end{tikzcd}$$

(we don't prove)

h: bilinear map there exist unique g such that

$$g\circ h = f$$

Y tensor product of V and W

denote $V\otimes W$. obviously we

V, W linear space and V^* W^* is dual space of corr.. to V and W to describe the construction more precisely, let us consider the tensor product of dual space V^* and W^*

$v^*\in V^*$ $w\in W^*$ $v^*\otimes w^*$ is defined by

$$v^*\otimes w^*(v,w) = v^*(v)\cdot w^*(w) = \langle v,v^*\rangle\langle w,w^*\rangle$$

ie. $v^*\otimes w^*\in L(V,W,F)$ $\otimes$ $V^*\times W^*\to L(V,W,F)$

since V and W are dual spaces of V^* and W^*, therefore we can analogously define the tensor $V\otimes W$

$$V\otimes W\subset L(V^*,W^*,F)$$

suppose $V^*\otimes W^*$ and $V\otimes W$ are reducible, which means any elements in $V^*\otimes W^*$ or $V\otimes W$ can be written in the form

$v^*\otimes w^*$ and $v\otimes w$.

Then we could choose basis $\{a^{*i}\}$ $\{b^{*j}\}$ or $\{a_i\}$ $\{b_j\}$

for any element in $V^*\otimes W^*$ $V\otimes W$ we have

$$v^*\otimes w^* = \sum_{ij} v^*(a_i)\, w^*(b_j)\, a^{*i}\otimes b^{*j}$$

It is easy to verify that $\{a^{*i} \otimes b^{*j}\}$ is a basis of $V^* \otimes W^*$.
Therefore $V^* \otimes W^*$ is an mn-dimensional vector space
~~[illegible]~~ by the define of reducible.

$$V^* \otimes W^* = L(V, W, F) \qquad V \otimes W = L(V^*, W^*, F)$$

a ~~[illegible]~~ very natural thought is that $V^* \otimes W^*$ and $V \otimes W$ is dual to each other if we define

$$(v \otimes w, v^* \otimes w^*) = \langle v, v^* \rangle \langle w, w^* \rangle$$

It is easy to verify $\langle\ \rangle$ ~~[illegible]~~ bilinear ...
Particularly:

$$\langle a_i \otimes b_j, a^{*p} \otimes b^{*q} \rangle = \delta_i^p \delta_j^q$$

therefore $V^* \otimes W^* = (V \otimes W)^*$ That's
~~That~~ ~~we could~~ The operation of the tensor product of linear functions can be generalized to arbitrary multilinear functions $f \in L(V_1 \cdots V_s, F)$ $g \in L(W_1, \cdots W_s, F)$

$$f \otimes g(v_1 \cdots v_s, w_1 \cdots w_s) = f(v_1 \cdots v_s) \cdot g(w_1 \cdots w_r)$$

$$f \otimes g \in L(V_1 \cdots V_s, W_1 \cdots W_s, F)$$

then we can define the operation of tensor

$$\varphi \otimes \psi \otimes \xi = (\varphi \otimes \psi) \otimes \xi = \varphi \otimes (\psi \otimes \xi)$$

Suppose our tensor spaces ~~is~~ are reducible
then we could define

$$V_1 \otimes V_2 \cdots \otimes V_s = L(V_1^*, V_2^* \cdots V_s^*, F)$$

now define $V_s^r = \underbrace{V \otimes V \cdots \otimes V}_{r \text{ terms}} \otimes \underbrace{V^* \otimes \cdots \otimes V^*}_{s \text{ terms}}$

is called (r, s) type-tensor where r is contravariant order s is covariant order.

obviously, $V_s^r = L(\underbrace{V^* \cdots V^*}_{r \text{ terms}}, \underbrace{V \cdots V}_{s \text{ terms}}; F)$

suppose $\{e_i\}$ $\{e^{*i}\}$ are dual bases in V and V^*

Date.

then (r,s)-type tensor x

$$x = \sum_{\substack{i_1 \cdots i_r \\ k_1 \cdots k_s}} x^{i_1 \cdots i_r}_{k_1 \cdots k_s} e_{i_1} \otimes \cdots \otimes e_{i_r} \otimes e^{*k_1} \otimes \cdots \otimes e^{*k_s}$$

where $x^{i_1 \cdots i_r}_{k_1 \cdots k_s} = x(e^{*i_1}, \cdots, e^{*i_r}, e_{k_1} \cdots e_{k_s})$

$$= \langle e^{*i_1} \otimes \cdots \otimes e^{*i_r} \otimes e_{k_1} \otimes \cdots \otimes e_{k_s}, x \rangle$$

now we use summation convention of Einstein; if an index occurs as both a subscript and superscript in the same term then the term is summed over the range of repeated index, and the summation sign is omitted.

$$x = x^{i_1 \cdots i_r}_{k_1 \cdots k_s} e_{i_1} \otimes \cdots \otimes e_{i_r} \otimes e^{*k_1} \otimes \cdots \otimes e^{*k_s}$$

then we consider when the basis of V is changed, then components of a tensor are changed. Suppose $\{\bar{e}_i\}$ $\{\bar{e}^{*i}\}$

$$\bar{e}_i = \alpha^j_i e_j$$

$$\bar{e}^{*k} = \beta^k_s e^{*s}$$

because $\langle \bar{e}_i, \bar{e}^{*k} \rangle = \delta^k_i$

$$\Rightarrow \delta^k_i = \alpha^j_i \beta^k_s \langle e_j, e^{*s} \rangle = \alpha^j_i \beta^k_s \delta^s_j$$

$$= \alpha^j_i \beta^k_j = \beta^k_j \alpha^j_i$$

~~there fore~~ therefore:

$$x = \bar{x}^{i_1 \cdots i_r}_{k_1 \cdots k_r} (\quad) = \bar{x}^{i_1 \cdots i_r}_{k_1 \cdots k_r} \alpha \cdots \beta (\quad ,$$

$$\Rightarrow x^{j_1 \cdots j_r}_{l_1 \cdots l_s} = \bar{x}^{i_1 \cdots i_r}_{k_1 \cdots k_s} \alpha^{j_1}_{i_1} \cdots \alpha^{j_r}_{i_r} \beta^{k_1}_{l_1} \cdots \beta^{k_s}_{l_s}$$

In classical tensor analysis, this is used to define tensor. samyly we could define the tensor operation between an (r_1, s_1)-type tensor x and an (r_2, s_2)-type tensor y. $x \otimes y$ is an (r_1+r_2, s_1+s_2)-type tensor given by

Date.

$$x \otimes y\,(v^{*r_1} \cdots v^{*r_1+r_2}, v_1 \cdots v_{s_1+s_2})$$
$$= x(\quad)\quad y(\quad)$$

Suppose reducible, when a basis is chosen, the component of $x \otimes y$

$$(x\otimes y)^{i_1 \cdots i_{r_1+r_2}}_{k_1 \cdots k_{s_1+s_2}} = x^{i_1 \cdots i_{r_1}}_{k_1 \cdots k_{s_1}} \cdot y^{i_{r_1+1} \cdots i_{r_1+r_2}}_{k_{s_1+1} \cdots k_{s_1+s_2}}$$

As we discussed.

Suppose $T^r(V) = V^r_0 = \underbrace{V \otimes \cdots \otimes V}_{r \text{ terms}}$

Now we define the direct sum $T(V) = \sum_{r \geq 0} T^r(V)$

Any element x in it 清有限项 can be expressed as $x = \sum_{r\geq 0} x^r \quad x^r \in T^r(V)$

where all but finitely many terms are zero. Thus $T(V)$ is an infinite-dimensional vector space, ~~whi~~ with the operation of tensor the vector space $T(V)$ becomes an algebra, and is called tensor algebra.

Simimarly, then tensor algebra of V^* is $T(V^*) = \sum_{r \geq 0} V^0_r$

obviously, as we discussed before, under reducible, $T(V^*)$ and $T(V)$ are dual to each other.

$$\langle v_1 \otimes \cdots \otimes v_r, v^{*1} \otimes \cdots \otimes v^{*r} \rangle = \langle v_1, v^{*1} \rangle \cdots \langle v_r, v^{*r} \rangle$$

Denote the permutation group of the set of natural numbers $\{1 \cdots r\}$ $S(r)$, $\sigma \in S(r)$ determines an automorphism of the vector space $T^r(V)$ suppose $x \in T^r(V)$, we define

$$\sigma x (v^{*1}, \cdots, v^{*r}) = x(v^{*\sigma(1)}, \cdots v^{*\sigma(r)})$$

It is easy to see that if $x = v_1 \otimes \cdots \otimes v_r$ then

$$\sigma x = v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(r)}$$

because $x(v^{*\sigma(1)} \cdots) = \langle v_1, v^{*\sigma(1)} \rangle \cdots \langle v_r, v^{*\sigma(r)} \rangle$

$$= \langle v_{\sigma^{-1}(1)}, v^{*1} \rangle \cdots$$

Def: Suppose $x \in T^r(V)$. If for any $\sigma \in S(r)$ we have.

$\cancel{\sigma(x)}$ $\sigma x = x$ $\quad$ x a symmetric contravariant ~~for~~ tensor of order r.

If for any $\sigma \in S(r)$ we have $\sigma(x) = \operatorname{sgn}\sigma \cdot x$ Date.

x alternating contravariant tensor of order.

obviously, A necessary and sufficient condition for x to be a symmetric tensor is that its components are symmetric with respect to all indices. same for alternating tensor.

$x^{i_1 \cdots i_r} = x(e^{*i_1} \cdots e^{*i_r}) = \sigma x(e^{*i_1} \cdots e^{*i_r}) = x^{i_{\sigma(1)} \cdots i_{\sigma(r)}}$

Denote the set of all symmetric contravariant tensor of order r by $P^r(V)$ and the set of all alternating contravariant tensor of order r by $A^r(V)$ ob. $P^r(V)$ $A^r(V)$ are linear subspaces of $T^r(V)$

Def:
$$S_r(x) = \frac{1}{r!} \sum_{\sigma \in S(r)} \sigma x$$

$$A_r(x) = \frac{1}{r!} \sum_{\sigma \in S(r)} \operatorname{sgn}\sigma \cdot \sigma x$$

Then we could easy ~~proved~~.

$$P^r(V) = S_r(T^r(V)) \quad \Lambda^r(V) = A_r(T^r(V)).$$

Exterior Algebra:

An alternating contravariant tensor of order r is also called an exterior vector of degree r or an exterior r-vector. The space $\Lambda^r(V)$ is called the exterior space of V of degree r.

let $A^1(V) = V$ $A^0(V) = \mathbb{F}$

More importantly, there exists an operation, the exterior product such that the product of two exterior vectors is another exterior vector.

suppose ξ is an exterior k-vector, and η an exterior l-vector.

$$\xi \wedge \eta = A_{l+k}(\xi \otimes \eta)$$

Date Then $\xi\wedge\eta$ is an exterior $(k+l)$-vector, called the exterior (wedge) product of ξ and η.

Theorem: The exterior product satisfy

Suppose $\xi, \xi_1, \xi_2 \in \Lambda^k(V)$, $\eta, \eta_1, \eta_2 \in \Lambda^l(V)$. $\zeta \in \Lambda^h(V)$

Then we have

(1). Distributive Law $(\xi_1+\xi_2)\wedge\eta = \xi_1\wedge\eta + \xi_2\wedge\eta$

$$\xi\wedge(\eta_1+\eta_2) = \xi\wedge\eta_1 + \xi\wedge\eta_2$$

(2) Anticommutative Law $\xi\wedge\eta = (-1)^{kl}\eta\wedge\xi$

(3) $(\xi\wedge\eta)\wedge\zeta = \xi\wedge(\eta\wedge\zeta)$

Associative.

Prof: (1) ob (2) ob (3) ob

suppose $\xi,\eta\in\Lambda^1(V)$ $\quad \xi\wedge\eta = -\eta\wedge\xi$

$$\xi\wedge\xi = \eta\wedge\eta = 0$$

satisfy the anticommutative law

suppose $\{e_1 \cdots e_n\}$ is a basis of V

Then according the associate law

$$e_{i_1}\wedge e_{i_2}\cdots\wedge e_{i_r} = A_r(e_{i_1}\otimes\cdots\otimes e_{i_r})$$

according to the Anticommutative, an exterior vector $e_{i_1}\wedge\cdots\wedge e_{i_r}$ is nonzero only if $i_1 \cdots i_r$ are distinct.

If $r>n$ then there must be reapted indices among $i_1\cdots i_r$.

Suppose ξ is an exterior r-vector, with the following

$$\xi = \xi^{i_1\cdots i_r} e_{i_1}\otimes\cdots\otimes e_{i_r}$$

$$\xi = A_r\xi = \xi^{i_1\cdots i_r} e_{i_1}\wedge\cdots\wedge e_{i_r} = 0 \quad \text{if } r>n.$$

suppose $r\le n$

$$\xi = r! \sum_{i_1<\cdots<i_r} \xi^{i_1\cdots i_r} e_{i_1}\wedge\cdots\wedge e_{i_r}$$

$r\le n$, $\{e_{i_1}\wedge\cdots\wedge e_{i_r}, 1\le i_1<\cdots<i_r\le n\}$ forms a basis of $\Lambda^r(V)$

prof: we only need show that these $\binom{n}{r} = \frac{n!}{r!(n-r)!}$ exterior vectors are linearly independent.

suppose $v^{*1}, \cdots, v^{*r}$ are r arbitrary elements of V^*.

$$e_{i_1} \wedge e_{i_2} \cdots \wedge e_{i_r}(v^{*1}, \cdots, v^{*r}) = \frac{1}{r!} \sum \operatorname{sgn}\sigma \langle e_{i_1}, v^{*\sigma(1)} \rangle \cdots \langle e_{i_r}, v^{*\sigma(r)} \rangle$$

$$= \frac{1}{r!} \begin{vmatrix} \langle e_{i_1}, v^{*1} \rangle & \cdots & \langle e_{i_1}, v^{*r} \rangle \\ \vdots & \ddots & \vdots \\ \langle e_{i_r}, v^{*1} \rangle & \cdots & \langle e_{i_r}, v^{*r} \rangle \end{vmatrix}$$

In particular. $e_{i_1} \wedge \cdots \wedge e_{i_r}(e^{*j_1}, \cdots e^{*j_r})$

$$= \frac{1}{r!} \det(\langle e_{i_\alpha}, e^{*j_\beta} \rangle)$$

$$= \frac{1}{r!} \delta^{j_1 \cdots j_r}_{i_1 \cdots i_r}$$

$$\delta^{j_1 \cdots j_r}_{i_1 \cdots i_r} = \begin{cases} 1 & \text{if } i_1 \cdots i_r \text{ distinct, and } \{j_1 \cdots j_r\} \text{ even p of } (i_1 \cdots) \\ -1 & \text{odd permutation of } (i_1 \cdots i_r) \\ 0 & \text{otherwise.} \end{cases}$$

For $r<n$ suppose

$$\sum_{1 \le i_1 < \cdots < i_r \le n} a^{i_1 \cdots i_r} e_{i_1} \wedge \cdots \wedge e_{i_r} = 0$$

assume $a^{j_1 \cdots j_r} \neq 0$ $1 \le j_1 < \cdots < j_r \le n$, with remaining index $k_1 < \cdots < k_{n-r}$, That is

$$a^{j_1 \cdots j_r} e_{j_1} \wedge \cdots e_{j_r} \wedge e_{k_1} \wedge \cdots \wedge e_{k_{n-r}} = 0$$

$$\Rightarrow a^{j_1 \cdots j_r} = 0$$ $\square$

Denote the formal sum $\sum_{r=0}^{n} \Lambda^r(V)$ by $\Lambda(V)$

$\Lambda(V)$ - 2^n dimensional vector space. Let

$$\xi = \sum_{r=0}^{n} \xi^r \quad \eta = \sum_{s=0}^{n} \eta^s$$

Define $$\xi \wedge \eta = \sum_{r,s=0}^{n} \xi^r \wedge \eta^s$$

Then $\Lambda(V)$ becomes an algebra with respect to the exterior product. is called exterior algebra or Grassman algebra.

Date The set $\{1, e_i (1 \le i \le n), e_{i_1} \wedge e_{i_2} \cdots\}$ basis of $\Lambda(V)$

Similarly, we have exterior algebra for the dual space V^*

$$\Lambda(V^*) = \sum_{0 \le r \le n} \Lambda^r(V^*)$$

$\Lambda^r(V)$ & $\Lambda^r(V^*)$ are dual to each other.

$$\langle v_1 \wedge \cdots \wedge v_r, v^{*1} \wedge \cdots \wedge v^{*r} \rangle = \det(\langle v_\alpha, v^{*\beta} \rangle)$$

Exterior Differentiation:

Suppose M is an m-dimensional smooth manifold. The bundle of exterior r-forms on M

$$\Lambda^r(M^*) = \bigcup_{p \in M} \Lambda^r(T_p^*)$$

$$A^r(M) = \Gamma(\Lambda^r(M^*))$$

$A^r(M)$ are called exterior differential r-forms.

Therefore, an exterior differential r-form on M is a smooth skew-symmetric covariant tensor field of order r on M.

$\Lambda(M^*) = \bigcup_{p \in M} \Lambda(T_p^*)$ the elements of the space of its sections $A(M)$ are called exterior differential forms on M.

$$A(M) = \sum_{r=0}^{m} A^r(M)$$

i.e every differential form w can be written as

$$w = w^0 + w^1 + \cdots + w^m \qquad w^i \in A^i(M)$$

The wedge product of exterior forms can be extended to the space of exterior differential forms $A(M)$. Suppose $w_1, w_2 \in A(M)$

For any $p \in M$, let $w_1 \wedge w_2(p) = w_1(p) \wedge w_2(p)$

Then the space $A(M)$ becomes an algebra with respect to ... called graded algebra

Under the local coordinates $u^1 \cdots u^m$, the restriction of the exterior differential r-forms w in the coordinate neighborhood U

$$w = a_{i_1, \cdots a_{i_r}} du^{i_1} \wedge \cdots \wedge du^{i_r}$$

where $a_{i_1 \cdots i_r}$ is a smooth function on U which is skew-symmetric with respect to the lower indices.

by the definition, we have

$$\left\langle \frac{\partial}{\partial u^{i_1}} \wedge \cdots \wedge \frac{\partial}{\partial u^{i_r}}, du^{j_1} \wedge \cdots \wedge du^{j_r} \right\rangle = \delta^{j_1 \cdots j_r}_{i_1 \cdots i_r}$$

The space $A(M)$ of exterior differential forms plays a crucial role in manifold theory, due to the existence of an exterior derivative operator d on $A(M)$ which gives zero on operating twice.

Theorem: Suppose M is an m-dimensional smooth manifold. Then there exists a unique map $d: A(M) \longrightarrow A(M)$ such that $d(A^r(M)) \subset A^{r+1}(M)$ and such that d satisfies the following

(1) For any $w_1 w_2 \in A(M)$, $d(w_1 + w_2) = dw_1 + dw_2$

(2) Suppose w_1 is an exterior differential r-form. Then

$$d(w_1 \wedge w_2) = dw_1 \wedge w_2 + (-1)^r w_1 \wedge dw_2$$

(3) If f is a smooth function on M, i.e. $f \in A^0(M)$, then df is precisely the differential of f.

(4) $d(df) = 0 \quad f \in A^0(M)$

First we show that if the exterior derivative operator d exists, then d is a local operator. we only need to show that $w|_U = 0$ implies $dw|_U = 0$. Choose any point $p \in U$. By local compactness of manifolds, there is an open neighborhood W contain p such that $p \in W \subset \overline{W} \subset U$, there exists a smooth function h on M s.t

$$h(p') = \begin{cases} 1 & p' \in W \\ 0 & p' \notin U \end{cases}$$

Thus $hw = 0$. Therefore we have

$$dh \wedge w + h dw = 0$$

$$dw|_{W} = 0 \quad \Rightarrow dw \text{ in } U \text{ must be zero}$$

Date

So dw is therefore well-defined.

Now we show the uniqueness of the exterior derivative d within a local coordinate neighborhood. We only need to show this for a monomial. Suppose in a coordinate neighborhood U, w is expressed by

$$w = a\, du^1 \wedge \cdots \wedge du^r$$

Because action of d on an exterior differential form satisfies (1)-(4)

Therefore dw must be

$$dw = da \wedge du^1 \wedge \cdots \wedge du^r$$

and obviously if $\alpha_1 = a\, du^{i_1} \wedge \cdots \wedge du^{i_r}$ $\alpha_2 = b\, du^{j_1} \wedge \cdots \wedge du^{j_s}$

$$d(\alpha_1 \wedge \alpha_2) = (b\,da + a\,db) \wedge du^{i_1} \wedge \cdots \wedge du^{j_s}$$

$$= (da \wedge du^{i_1} \wedge \cdots \wedge du^{i_r}) \wedge (b\, du^{j_1} \wedge \cdots \wedge du^{j_s}) +$$

$$(-1)^r (a\, du^{i_1} \wedge \cdots \wedge du^{i_r}) \wedge (db \wedge du^{j_1} \wedge \cdots \wedge du^{j_s})$$

$$= d\alpha_1 \wedge \alpha_2 + (-1)^r \alpha_1 \wedge d\alpha_2$$

(4) f smooth function on M

$$df = \frac{\partial f}{\partial u^i} du^i$$

since $f \in C^\infty(M)$ $\quad \frac{\partial^2 f}{\partial u^i \partial u^j} = \frac{\partial^2 f}{\partial u^j \partial u^i}$

$$d(df) = d\frac{\partial f}{\partial u^i} \wedge du^i$$

$$= \frac{\partial^2 f}{\partial u^i \partial u^j} du^j \wedge du^i$$

$$= \frac{1}{2}\left(\frac{\partial^2 f}{\partial u^i \partial u^j} - \frac{\partial^2 f}{\partial u^j \partial u^i}\right) du^i \wedge du^j = 0$$

Example:

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz$$

suppose the Cartesian coordinates in $\mathbb{R}^3$ (x, y, z)

$$\text{grad} f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$$

Suppose $\quad a = A\,dx + B\,dy + C\,dz$

$$da = \left(\frac{\partial C}{\partial y} - \frac{\partial B}{\partial z}\right) dy \wedge dz + \left(\frac{\partial A}{\partial z} - \frac{\partial C}{\partial x}\right) dz \wedge dx$$

$$+ \left(\frac{\partial B}{\partial x} - \frac{\partial A}{\partial y}\right) dx \wedge dy$$

Suppose $\alpha = A\,dy\wedge dz + B\,dz\wedge dx + C\,dx\wedge dy$

$\Rightarrow d\alpha = \left(\frac{\partial A}{\partial x} + \frac{\partial B}{\partial y} + \frac{\partial C}{\partial z}\right) dx\wedge dy\wedge dz$

By the theme $d^2=0 \Rightarrow \begin{cases} \operatorname{curl}(\operatorname{grad} f)=0 \\ \operatorname{div}(\operatorname{curl}(X))=0 \end{cases}$

Integrals of Differential forms:

The simplest device connecting the local and global properties of a manifold is the integration of exterior differential forms on the manifold.

Defination: An m-dimensional smooth manifold M is called orientable if there exists a continuous and nonvanishing exterior differential m-form ω on M. If M is given such an ω, then M is said to be oriented. If two such forms are given on M such that they differ by a function factor which is always positive, then we say that they assign the same orientation to M

"orientation", This ~~world~~ word stems from the Latin oriens ("east") and originally ~~meat~~ meant "turning toward the east" or more generally "positioning with respect to one's surroundings."

we begin with orientations of vector spaces we are all familiar with certain informal rules for ~~sing~~ singling out preferred ordered base of $\mathbb{R}^1$ $\mathbb{R}^2$ $\mathbb{R}^3$ we usually choose a basis for $\mathbb{R}^1$ that points to the right. And in $\mathbb{R}^2$ counterclockwise And in $\mathbb{R}^3$ about "right-handed" bases in $\mathbb{R}^3$ and "left-handed" And now we should translate these rules for selecting preferred bases of $\mathbb{R}^1$ $\mathbb{R}^2$ $\mathbb{R}^3$ into rigorous mathematical lanuage.

Define we say that two ordered bases $(E_1,\cdots,E_n)$ and $(\tilde{E}_1,\cdots,\tilde{E}_n)$ for V are consistently oriented if ~~there~~ the matrix (B_i^j) defined by

$$E_i = B_i^j \tilde{E}_j$$

A pointwise orientation is said to be ***continuous*** if every point of M is in the domain of an oriented local frame. (Recall that by definition the vector fields that make up a local frame are continuous.) An ***orientation of*** M is a continuous pointwise orientation. We say that M is ***orientable*** if there exists an orientation for it, and ***nonorientable*** if not. An ***oriented manifold*** is an ordered pair $(M, \mathcal{O})$, where M is an orientable smooth manifold and $\mathcal{O}$ is a choice of orientation for M; an ***oriented manifold with boundary*** is defined similarly. For each $p \in M$, the orientation of T_pM determined by $\mathcal{O}$ is denoted by $\mathcal{O}_p$. When it is not important to name the orientation explicitly, we use the usual shorthand expression "M is an oriented smooth manifold" (or "manifold with boundary").

Proposition 15.5 (The Orientation Determined by an n-Form). *Let M be a smooth n-manifold with or without boundary. Any nonvanishing n-form ω on M determines a unique orientation of M for which ω is positively oriented at each point. Conversely, if M is given an orientation, then there is a smooth nonvanishing n-form on M that is positively oriented at each point.*

Proposition 15.21. *Suppose M is an oriented smooth n-manifold with or without boundary, S is an immersed hypersurface with or without boundary in M, and N is*

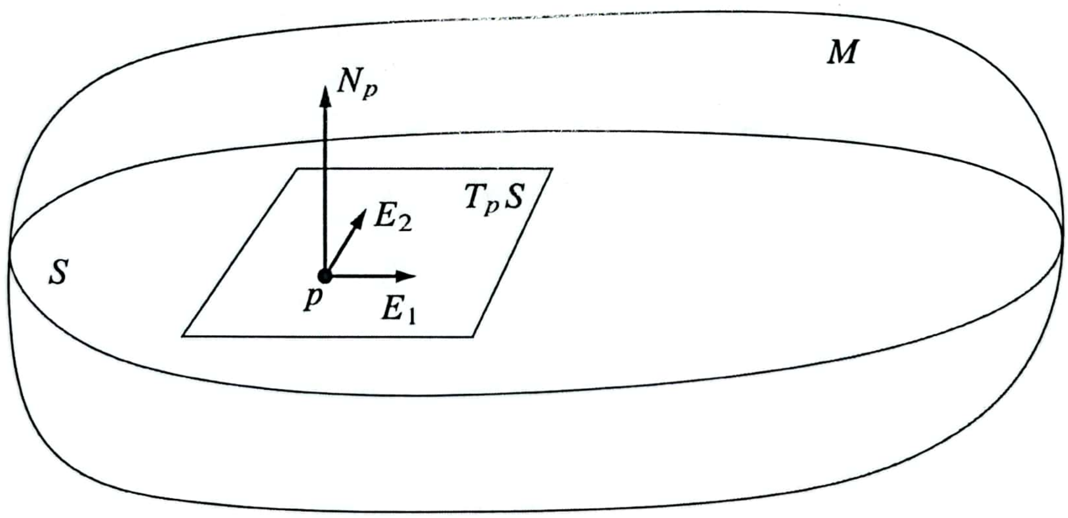


Fig. 15.5 The orientation induced by a nowhere tangent vector field

a vector field along S that is nowhere tangent to S. Then S has a unique orientation such that for each $p \in S$, $(E_1, \dots, E_{n-1})$ is an oriented basis for T_pS if and only if $(N_p, E_1, \dots, E_{n-1})$ is an oriented basis for T_pM. If ω is an orientation form for M, then $\iota_S^*(N \lrcorner\, \omega)$ is an orientation form for S with respect to this orientation, where $\iota_S \colon S \hookrightarrow M$ is inclusion.

Theorem 3.2 (Partition of Unity Theorem). *Suppose Σ is an open covering of a smooth manifold M. Then there exists a family of smooth functions $\{g_\alpha\}$ on M satisfying the following conditions:*

1) $0 \leq g_\alpha \leq 1$, *and* supp g_α *is compact for each* α. *Moreover, there exists an open set* $W_i \in \Sigma$ *such that* supp $g_\alpha \subset W_i$*;*
2) *For each point* $p \in M$, *there is a neighborhood* U *that intersects* supp g_α *for only finitely many* α*;*
3) $\sum_\alpha g_\alpha = 1$.

With the above background, we can proceed to define the integration of exterior differential forms on a manifold M. Suppose M is an m-dimensional smooth manifold, and φ is an exterior differential m-form on M with a compact support. Choose any coordinate covering $\Sigma = \{W_i\}$ which is consistent with the orientation of M, and suppose that $\{g_\alpha\}$ is a partition of unity subordinate to Σ. Then

$$\varphi = \left(\sum_\alpha g_\alpha\right) \cdot \varphi = \sum_\alpha (g_\alpha \cdot \varphi). \tag{3.15}$$

Clearly, supp $(g_\alpha \cdot \varphi) \subset$ supp g_α is contained in some coordinate neighborhood $W_i \in \Sigma$. Therefore we can define

$$\int_M g_\alpha \cdot \varphi = \int_{W_i} g_\alpha \cdot \varphi, \tag{3.16}$$

where the right hand side is the usual Riemann integral, that is, if $g_\alpha \cdot \varphi$ with respect to the coordinate system $u^1, \dots, u^m$ in W_i is expressed as

$$f(u^1, \dots, u^m) du^1 \wedge \cdots \wedge du^m,$$

then the integral on the right hand side in (3.16) is [redacted]

$$\int_{W_i} f(u^1, \dots, u^m) du^1 \cdots du^m. \tag{3.17}$$

To show that (3.16) is well-defined, we need only show that the right hand side is independent of the choice of W_i. Suppose supp $(g_\alpha \cdot \varphi)$ is contained in two coordinate neighborhoods W_i and W_j, and suppose the local coordinates consistent with the orientation of M are u^k and v^k, respectively. Then the Jacobian of the change of coordinates satisfies

$$J = \frac{\partial(v^1, \dots, v^m)}{\partial(u^1, \dots, u^m)} > 0. \tag{3.18}$$

and supp f = supp f' = supp $(g_\alpha \cdot \varphi) \subset W_i \cap W_j$. By the formula for the change of variables in the Riemann integral, we have

$$\begin{aligned}
\int\limits_{W_i \cap W_j} f' dv^1 \cdots dv^m &= \int\limits_{W_i \cap W_j} f' |J| du^1 \cdots du^m \\
&= \int\limits_{W_i \cap W_j} f \, du^1 \cdots du^m,
\end{aligned}$$

i.e.,

$$\int\limits_{W_i} g_\alpha \cdot \varphi = \int\limits_{W_j} g_\alpha \cdot \varphi. \tag{3.21}$$

Since supp φ is compact, it only intersects finitely many supp g_α by condition 2) of the Partition of Unity Theorem. Therefore the right hand side of (3.15) is a sum of only finitely many terms. Let

$$\int\limits_M \varphi = \sum_\alpha \int\limits_M g_\alpha \cdot \varphi. \tag{3.22}$$

For any given partition of unity $\{g_\alpha\}$ subordinate to Σ, the right hand side of (3.22) is completely determined. Now we show that (3.22) is independent of the choice of the partition of unity $\{g_\alpha\}$.

Suppose $\{g'_\beta\}$ is another partition of unity subordinate to Σ. Then

$$\begin{aligned}
\sum_\beta \int\limits_M g'_\beta \cdot \varphi &= \sum_\beta \sum_\alpha \int\limits_M g_\alpha \cdot g'_\beta \cdot \varphi \\
&= \sum_\alpha \int\limits_M \sum_\beta g'_\beta \cdot g_\alpha \cdot \varphi \\
&= \sum \int g_\alpha \cdot \varphi.
\end{aligned}$$

Definition 3.4. Suppose M is an m-dimensional oriented smooth manifold and φ is an exterior differential m-form on M with compact support. The numerical value $\int\limits_M \varphi$ defined in (3.22) is called the **integral** of the exterior differential form φ on M.

Definition 4.1. Suppose M is an m-dimensional smooth manifold. A **region D with boundary** is a subset of the manifold M with two kinds of points:

1) *Interior points*, each of which has a neighborhood in M contained in D.
2) *Boundary points* p, for each of which there exists a coordinate chart $(U; u^i)$ such that $u^i(p) = 0$ and

$$U \cap D = \{q \in U | u^m(q) \geq 0\}. \tag{4.8}$$

A coordinate system u^i with the above property is called an **adapted coordinate system** for the boundary point p.

The set of all the boundary points of D is called the **boundary** of D, denoted by B.

Theorem 4.1. *The boundary B of a region D with boundary is a regular imbedded closed submanifold. If M is orientable, then B is also orientable.*

Proof. The boundary B of the region D is obviously a closed subset of M. Suppose $(U; u^i)$ is an adapted coordinate neighborhood. Then

$$U \cap B = \{q \in U | u^m(q) = 0\}. \tag{4.9}$$

By Definition 3.2 in Chapter 1, B is a regular imbedded closed submanifold of M.

Suppose M is an oriented manifold. Choose an adapted coordinate neighborhood $(U; u^i)$ which is consistent with the orientation of M at an arbitrary point $p \in B$. Then $(u, \dots, u^{m-1})$ is a local coordinate system of B at the point p. Let

$$(-1)^m du^1 \wedge \cdots \wedge du^{m-1} \tag{4.10}$$

specify the orientation of the boundary B in the coordinate neighborhood $U \cap B$ of the point p. We will prove that the orientations given in this way to the coordinate neighborhoods are consistent. Suppose $(V; v^i)$ is another coordinate neighborhood of a boundary point p consistent with the orientation of M. Then

$$\frac{\partial(v^1, \dots, v^m)}{\partial(u^1, \dots, u^m)} > 0. \tag{4.11}$$

Suppose $v^m = f^m(u^1, \dots, u^m)$. Then for any fixed $u^i, \dots, u^{m-1}$ the sign of the variable v^m is the same as that of u^m, and $v^m = 0$ when $u^m = 0$. Therefore, at the point p, $\partial v^m / \partial u^m > 0$. Without loss of generality, we may assume that $v^m = u^m$. Then (4.11) becomes

$$\frac{\partial(v^1, \dots, v^{m-1})}{\partial(u^1, \dots, u^{m-1})} > 0. \tag{4.12}$$

This shows that $(-1)^m du^1 \wedge \cdots \wedge du^{m-1}$ and $(-1)^m dv^1 \wedge \cdots \wedge dv^{m-1}$ give consistent orientations in $U \cap V \cap B$. Hence B is orientable. $\square$

The orientation of B given in (4.10) is called the **induced orientation** on the boundary B by an oriented manifold M. If D has the same orientation as M we denote the boundary B with the induced orientation by ∂D. It is easy to verify that the orientations of ∂D and $\partial \Sigma$ in the preceding four examples are induced in this way.

Theorem 4.2 (Stokes' Formula). *Suppose D is a region with boundary in an m-dimensional oriented manifold M, and ω is an exterior differential $(m-1)$-form on M with compact support. Then*

$$\int_D d\omega = \int_{\partial D} \omega. \tag{4.13}$$

If $\partial D = \varnothing$, then the integral on the right hand side is zero.

Definition 1.1. Suppose E, M are two smooth manifolds, and $\pi : E \longrightarrow M$ is a smooth surjective map. Let $V = \mathbb{R}^q$ be a q-dimensional vector space. If an open covering $\{U, W, Z, \dots\}$ of M and a set of maps $\{\varphi_U, \varphi_W, \varphi_Z, \cdots\}$ satisfy all of the following conditions, then (E, M, π) is called a (real) q-dimensional **vector bundle** on M, where E is called the **bundle space**, M is called the **base space**, π is called the **bundle projection**, and $V = \mathbb{R}^q$ is called the **typical fiber**:

1) Every map φ_U is a diffeomorphism from $U \times \mathbb{R}^q$ to $\pi^{-1}(U)$, and for any $p \in U$, $y \in \mathbb{R}^q$,
$$\pi \circ \varphi_U(p, y) = p. \tag{1.21}$$
2) For any fixed $p \in U$, let
$$\varphi_{U,p}(y) = \varphi_U(p, y), \qquad y \in \mathbb{R}^q. \tag{1.22}$$
Then $\varphi_{U,p} : \mathbb{R}^q \longrightarrow \pi^{-1}(p)$ is a homeomorphism. When $U \cap W \neq \varnothing$, for any $p \in U \cap W$,
$$g_{UW}(p) = \varphi_{W,p}^{-1} \circ \varphi_{U,p} : \mathbb{R}^q \longrightarrow \mathbb{R}^q \tag{1.23}$$
is a linear automorphism of $V = \mathbb{R}^q$, i.e., $g_{UW}(p) \in GL(V)$.
3) When $U \cap W \neq \varnothing$, the map $g_{UW} : U \cap W \longrightarrow GL(V)$ is smooth.

Date

Consider these sense. how we want to add some structure on a manifold. ~~we~~ what method would use.

may be ~~most~~ the method we use is the best way to induce some structure on a manifold.

$$\begin{array}{ccc} U\times \mathbb{R}^q & \xrightarrow{\varphi_U} & E \\ & & \pi \downarrow \uparrow \alpha \\ & & M \end{array} \qquad \pi\circ\alpha = id_M$$

E. bundle space π. bundle projection

$V = \mathbb{R}^q$ is called typical fiber.

and the s is the most important structure.

s is called a smooth section.

And obviously, we could notice that E have the structure of $V = \mathbb{R}^q$ and the structure U.

if we fixed p the point in smooth manifold

consider ~~the~~ these section s, s_1, s_2

notice that E have the structure of V

so it's natural that

$$(s_1+s_2)(p) = s_1(p)+s_2(p)$$
$$(\alpha s)(p) = \alpha(p)\cdot s(p)$$

~~det~~ denote $\Gamma(E)$ is the set of section.

~~the~~ ~~consider~~ obviously, we already add the struct of V

§4–1 Connections on Vector Bundles

We have already described the concepts of a vector bundle and a section in §3–1. Suppose E is a q-dimensional real vector bundle on M, and $\Gamma(E)$ is the set of smooth sections of E on M. $\Gamma(E)$ is a real vector space; it is also a $C^\infty(M)$-module.

Definition 1.1. A **connection** on a vector bundle E is a map

$$D : \Gamma(E) \longrightarrow \Gamma(T^*(M) \otimes E), \tag{1.1}$$

which satisfies the following conditions:

1) For any $s_1, s_2 \in \Gamma(E)$,

$$D(s_1 + s_2) = Ds_1 + Ds_2.$$

2) For $s \in \Gamma(E)$ and any $\alpha \in C^\infty(M)$,

$$D(\alpha s) = d\alpha \otimes s + \alpha Ds.$$

Suppose X is a smooth tangent vector field on M and $s \in \Gamma(E)$. Let

$$D_X s = \langle X \, , \, Ds \rangle \, , \tag{1.2}$$

where $\langle \, , \, \rangle$ represents the pairing between $T(M)$ and $T^*(M)$. Then $D_X s$ is a section of E, called the **absolute differential quotient** or the **covariant derivative** of the section s along X.

Remark 2. D is an operator on sections of E, but it also has local properties. If s_1 and s_2 are two sections of E, and the restrictions of s_1 and s_2 to an open set U of M are the same, then the restrictions of Ds_1 and Ds_2 to U are also the same. The proof of this resembles the proof of the local properties of d (Theorem 2.1 of Chapter 3). Using the linearity of D, we need only show that if the restriction of a section s to an open set $U \subset M$ is zero, then $Ds|_U = 0$.

Remark 3. By (1.2), D as a bivalent map is an operator from $\Gamma(T(M)) \times \Gamma(E)$ to $\Gamma(E)$. It satisfies the following properties. Suppose X, Y are any two smooth tangent vector fields on M, s, s_1, s_2 are sections of E, and $\alpha \in C^\infty(M)$. Then

1) $D_{X+Y} s = D_X s + D_Y s$;
2) $D_{\alpha X} s = \alpha D_X s$;
3) $D_X(s_1 + s_2) = D_X s_1 + D_X s_2$;
4) $D_X(\alpha s) = (X\alpha)s + \alpha D_X s$.

As described in Remark 2, absolute differential quotient operators have the following local properties:

1) If X_1, X_2 are two tangent vector fields with the same value at a point p in M, then for any section s of E, $D_{X_1}s$ and $D_{X_2}s$ also have the same value at p. From this, we can define the absolute differential quotient of a section of E with respect to a tangent vector of M at point p. For $X \in T_p(M)$, D_X is a map from $\Gamma(E)$ to E_p.
2) For a map $D_X : \Gamma(E) \longrightarrow E_p (X \in T_p(M))$, $D_X s_1 = D_X s_2$ if the values of the sections s_1, s_2 on a parametrized curve in M that is tangent to X are the same.

The proofs of these local properties are similar to those for Remark 2. The reader may fill in the details.

Locally, a connection is given by a set of differential 1-forms. Suppose U is a coordinate neighborhood of M with local coordinates u^i, $1 \leq i \leq m$. Choose q smooth sections $s_\alpha (1 \leq \alpha \leq q)$ of E on U such that they are linearly independent everywhere. Such a set of q sections is called a **local frame field** of E on U. It is obvious that at every point $p \in U$, $\{du^i \otimes s_\alpha, 1 \leq i \leq m, 1 \leq \alpha \leq q\}$ forms a basis for the tensor space $T_p^* \otimes E_p$.

Because Ds_α is a local section on U of the bundle $T^*(M) \otimes E$, we can write

$$Ds_\alpha = \sum_{1 \leq i \leq m, 1 \leq \beta \leq q} \Gamma_{\alpha i}^{\beta} du^i \otimes s_\beta, \tag{1.5}$$

where $\Gamma_{\alpha i}^{\beta}$ are smooth functions on U. Denote

$$\omega_\alpha^\beta = \sum_{1 \leq i \leq m} \Gamma_{\alpha i}^{\beta} du^i. \tag{1.6}$$

Then (1.5) becomes

$$Ds_\alpha = \sum_{\beta=1}^{q} \omega_\alpha^\beta \otimes s_\beta. \tag{1.7}$$

Proof. Choose a coordinate neighborhood $(U; u^i)$ of p such that $u^i(p) = 0$, $1 \leq i \leq m$. Suppose S' is a local frame field on U with corresponding connection matrix $\omega' = (\omega'^{\beta}_{\alpha})$, where

$$\omega'^{\beta}_{\alpha} = \sum_{i=1}^{m} \Gamma'^{\beta}_{\alpha i} u^i, \tag{1.23}$$

and the $\Gamma'^{\beta}_{\alpha i}$ are smooth functions on U. Let

$$a^{\beta}_{\alpha} = \delta^{\beta}_{\alpha} - \sum_{i=1}^{m} \Gamma'^{\beta}_{\alpha i}(p) \cdot u^i. \tag{1.24}$$

Then $A = (a^{\beta}_{\alpha})$ is the identity matrix at p. Hence there exists a neighborhood $V \subset U$ of p such that A is nondegenerate in V. Thus

$$S = A \cdot S' \tag{1.25}$$

is a local frame field on V. Since

$$dA(p) = -\omega'(p),$$

we can obtain from (1.12) that

$$\begin{aligned} \omega(p) &= (dA \cdot A^{-1} + A \cdot \omega' \cdot A^{-1})(p) \\ &= -\omega'(p) + \omega'(p) = 0. \end{aligned} \tag{1.26}$$

Thus S is the desired local frame field. □

Definition 1.3. Suppose C is a parametrized curve in M, and X is a tangent vector field along C. If a section s of the vector bundle E on C satisfies

$$D_X s = 0, \tag{1.40}$$

then we say s is **parallel** along the curve C.

Suppose the curve C is given in a local coordinate neighborhood U of M by

$$u^i = u^i(t), \qquad 1 \le i \le m. \tag{1.41}$$

Then the tangent vector field of C is

$$X = \sum_{i=1}^{m} \frac{du^i}{dt} \frac{\partial}{\partial u^i}.$$

Let S be a local frame field on U. Then $s = \sum_{\alpha=1}^{q} \lambda^\alpha s_\alpha$ is a parallel section along C if and only if it satisfies the following system of equations:

$$\langle X\,,\, Ds\rangle = \sum_{\alpha=1}^{q} \left(\frac{d\lambda^\alpha}{dt} + \sum_{\beta,i} \Gamma^\alpha_{\beta i} \frac{du^i}{dt} \lambda^\beta \right) s_\alpha = 0,$$

that is,

$$\frac{d\lambda^\alpha}{dt} + \sum_{\beta,i} \Gamma^\alpha_{\beta i} \frac{du^i}{dt} \lambda^\beta = 0, \qquad 1 \le \alpha \le q. \tag{1.42}$$

Since (1.42) is a system of ordinary differential equations, a unique solution exists for any given initial values. Thus we see that if any vector $v \in E_p$ is given at a point p on C, then it determines uniquely a vector field parallel along C, which is called the **parallel displacement** of v along C. Obviously, the parallel displacements along C introduce isomorphisms among the fibers of the vector bundle E at different points on C.

A connection D of the vector bundle E induces a connection (also denoted by D) on the dual bundle E^*. Suppose $s \in \Gamma(E)$, $s^* \in \Gamma(E^*)$, and the pairing $\langle s\,,\, s^*\rangle$ is a smooth function on M. Then the induced connection D on E^* is determined by the equation

$$d\langle s\,,\, s^*\rangle = \langle Ds\,,\, s^*\rangle + \langle s\,,\, Ds^*\rangle\,, \tag{1.43}$$

where the notation $\langle\,,\,\rangle$ on the right hand side still means the pairing between E and E^*.

Let us find the matrix of the induced connection on E^*. Suppose $s_\alpha (1 \leq \alpha \leq q)$ is a local frame field on E, and the dual local frame field on E^* is $s^{*\beta} (1 \leq \beta \leq q)$, i.e.

$$\langle s_\alpha , s^{*\beta} \rangle = \delta_\alpha^\beta. \tag{1.44}$$

Let

$$Ds^{*\beta} = \sum_{\gamma=1}^{q} \omega_\gamma^{*\beta} \otimes s^{*\gamma}. \tag{1.45}$$

Then from (1.43) we have

$$\begin{aligned} \omega_\alpha^\beta &= \langle Ds_\alpha , s^{*\beta} \rangle = - \langle s_\alpha , Ds^{*\beta} \rangle \\ &= -\omega_\alpha^{*\beta}. \end{aligned}$$

Thus

$$Ds^{*\beta} = - \sum_{\alpha=1}^{q} \omega_\alpha^\beta \otimes s^{*\alpha}. \tag{1.46}$$

If the section s^* of E^* is expressed locally as

$$s^* = \sum_{\alpha=1}^{q} x_\alpha s^{*\alpha},$$

then from (1.46) we obtain

$$Ds^* = \sum_{\alpha=1}^{q} \left(dx_\alpha - \sum_{\beta=1}^{q} x_\beta \omega_\alpha^\beta \right) \otimes s^{*\alpha}. \tag{1.47}$$

$$D(s_1 \oplus s_2) = Ds_1 \oplus Ss_2, \tag{1.48}$$

$$D(s_1 \otimes s_2) = Ds_1 \otimes s_2 + s_1 \otimes Ds_2. \tag{1.49}$$

Then these equations determine connections on $E_1 \oplus E_2$ and $E_1 \otimes E_2$, respectively called the **induced connections** on $E_1 \oplus E_2$ and $E_1 \otimes E_2$.

Suppose t is a $(2,1)$-type tensor field expressed as

$$t = t^{ij}_k du^k \otimes \frac{\partial}{\partial u^i} \otimes \frac{\partial}{\partial u^j} \tag{2.14}$$

under the local coordinates u^i. From (1.49) in we have

$$\begin{aligned} Dt &= dt^{ij}_k \otimes du^k \otimes \frac{\partial}{\partial u^i} \otimes \frac{\partial}{\partial u^j} + t^{ij}_k D(du^k) \otimes \frac{\partial}{\partial u^i} \otimes \frac{\partial}{\partial u^j} \\ &\quad + t^{ij}_k du^k \otimes D\left(\frac{\partial}{\partial u^i}\right) \otimes \frac{\partial}{\partial u^j} + t^{ij}_k du^k \otimes \frac{\partial}{\partial u^i} \otimes D\left(\frac{\partial}{\partial u^j}\right) \\ &= \left(dt^{ij}_k - t^{ij}_l \omega^l_k + t^{lj}_k \omega^i_l + t^{il}_k \omega^j_l\right) \otimes du^k \otimes \frac{\partial}{\partial u^i} \otimes \frac{\partial}{\partial u^j} \\ &= t^{ij}_{k,h} du^h \otimes du^k \otimes \frac{\partial}{\partial u^i} \otimes \frac{\partial}{\partial u^j}, \end{aligned} \tag{2.15}$$

where

$$t^{ij}_{k,h} = \frac{\partial t^{ij}_k}{\partial u^h} - t^{ij}_l \Gamma^l_{kh} + t^{lj}_k \Gamma^i_{lh} + t^{il}_k \Gamma^j_{lh}. \tag{2.16}$$

The absolute differential of a scalar field is defined to be its ordinary differential.

Definition 2.1. Suppose $C : u^i = u^i(t)$ is a parametrized curve on M, and $X(t)$ is a tangent vector field defined on C given by

$$X(t) = x^i(t) \left(\frac{\partial}{\partial u^i}\right)_{C(t)}. \tag{2.17}$$

We say that $X(t)$ is **parallel** along C if its absolute differential along C is zero, i.e., if

$$\frac{DX}{dt} = 0. \tag{2.18}$$

If the tangent vectors of a curve C are parallel along C, then we call C a **self-parallel curve**, or a **geodesic**.

Equation (2.18) is equivalent to

$$\frac{dx^i}{dt} + x^j \Gamma^i_{jk} \frac{du^k}{dt} = 0. \tag{2.19}$$

This is a system of first-order ordinary differential equations. Thus a given tangent vector X at any point on C gives rise to a parallel tangent vector field, called the **parallel displacement** of X along the curve C. By the general discussion in §4–1, we see that a parallel displacement along C establishes an isomorphism between the tangent spaces at any two points on C.

If C is a geodesic, then its tangent vector

$$X(t) = \frac{du^i(t)}{dt} \left(\frac{\partial}{\partial u^i}\right)_{C(t)}$$

is parallel along C. Therefore a geodesic curve C should satisfy:

$$\frac{d^2 u^i}{dt^2} + \Gamma^i_{jk} \frac{du^j}{dt} \frac{du^k}{dt} = 0. \tag{2.20}$$

This is a system of second-order ordinary differential equations. Thus there exists a unique geodesic through a given point of M which is tangent to a given tangent vector at that point.

then (2.5) implies

$$T'^j_{ik} = T^q_{pr} \frac{\partial w^j}{\partial u^q} \frac{\partial u^p}{\partial w^i} \frac{\partial u^r}{\partial w^k}. \tag{2.32}$$

Hence T^j_{ik} satisfies the transformation rule for the components of $(1,2)$-type tensors. Thus

$$T = T^j_{ik} \frac{\partial}{\partial u^j} \otimes du^i \otimes du^k \tag{2.33}$$

is a $(1,2)$-type tensor, called the **torsion tensor** of the affine connection D. By (2.31) the components of the torsion tensor T are skew-symmetric with respect to the lower indices, that is,

$$T^j_{ik} = -T^j_{ki}. \tag{2.34}$$

Definition 2.2. If the torsion tensor of an affine connection D is zero, then the connection is said to be **torsion-free.**

A torsion-free affine connection always exists. In fact, if the coefficients of a connection D are Γ^j_{ik}, then set

$$\tilde{\Gamma}^j_{ik} = \frac{1}{2}\left(\Gamma^j_{ik} + \Gamma^j_{ki}\right). \tag{2.37}$$

Obviously, $\tilde{\Gamma}^j_{ik}$ is symmetric with respect to the lower indices and satisfies (2.5) under a local change of coordinates. Therefore the $\tilde{\Gamma}^j_{ik}$ are the coefficients of some connection $\tilde{D}$, and $\tilde{D}$ is torsion-free.

Any connection can be decomposed into a sum of a multiple of its torsion tensor and a torsion-free connection. In fact, (2.31) and (2.37) give

$$\Gamma^j_{ik} = -\frac{1}{2}T^j_{ik} + \tilde{\Gamma}^j_{ik}, \tag{2.38}$$

that is,

$$D_X Z = \frac{1}{2}T(X,Z) + \tilde{D}_X Z. \tag{2.39}$$

The geodesic equation (2.20) is equivalent to

$$\frac{d^2u^i}{dt^2} + \tilde{\Gamma}^i_{jk} \frac{du^j}{dt} \frac{du^k}{dt} = 0. \tag{2.40}$$

Thus a connection D and the corresponding torsion-free connection $\tilde{D}$ have the same geodesics.

Theorem 2.1. *Suppose D is a torsion-free affine connection on M. Then for any point $p \in M$ there exists a local coordinate system u^i such that the corresponding connection coefficients Γ^j_{ik} vanish at p.*

Proof. Suppose $(W; w^i)$ is a local coordinate system at p with connection coefficients Γ'^{j}_{ik}. Let

$$u^i = w^i + \frac{1}{2}\Gamma'^{i}_{jk}(p)\left(w^j - w^j(p)\right)\left(w^k - w^k(p)\right). \tag{2.41}$$

Then

$$\left.\frac{\partial u^i}{\partial w^j}\right|_p = \delta^i_j, \qquad \left.\frac{\partial^2 u^i}{\partial w^j \partial w^k}\right|_p = \Gamma'^{i}_{jk}(p). \tag{2.42}$$

Thus the matrix $\left(\dfrac{\partial u^i}{\partial w^j}\right)$ is nondegenerate near p, and (2.41) provides for a change of local coordinates in a neighborhood of p. From (2.5) we see that the connection coefficients Γ^j_{ik} in the new coordinate system u^i satisfy

$$\Gamma^j_{ik}(p) = 0, \qquad 1 \le i, j, k \le m.$$

Definition 2.1. Suppose M is an m-dimensional Riemannian manifold. If a parametrized curve C is a geodesic curve in M with respect to the Levi-Civita connection, then C is called a **geodesic** of the Riemannian manifold M.

Suppose the coefficients of the Levi-Civita connection D under the local coordinates u^i are Γ^i_{jk}. Then the curve $C : u^i = u^i(t)$ $(1 \le i \le m)$ is a geodesic if it satisfies the following second order differential equation:

$$\frac{d^2u^i}{dt^2} + \Gamma^i_{jk}\frac{du^j}{dt}\frac{du^k}{dt} = 0. \tag{2.1}$$

Here $X^i = du^i/dt$ is a tangent vector of C. By definition, the tangent vector of a geodesic is parallel along the curve itself with respect to the Levi-Civita connection, which also preserves metric properties under parallel displacement. Therefore the length of the tangent vector X^i of a geodesic is constant, that is,

$$g_{ij}\frac{du^i}{dt}\frac{du^j}{dt} = \text{const.},$$

or

$$\frac{ds}{dt} = \text{const.} \tag{2.2}$$

Hence we see that the parameter for a geodesic curve in a Riemannian manifold must be a linear function of the arc length s, i.e.,

$$t = \lambda s + \mu, \tag{2.3}$$

where $\lambda\,(\neq 0)$ and μ are constants.

We now consider a special coordinate system near a point such that the coordinates of any geodesic starting from that point are linear functions of the arc length. First we discuss this under the general assumption that M is an affine connection space.

Suppose the equation of a geodesic under the coordinates system $(U; u^i)$ is

$$\frac{d^2u^i}{dt^2} + \Gamma^i_{jk}\frac{du^j}{dt}\frac{du^k}{dt} = 0. \tag{2.4}$$

By the theory of ordinary differential equations, there exist for any point $x_0 \in U$ a neighborhood $W \subset U$ of x_0 and positive numbers r, δ such that for any initial value $x \in W$ and $\alpha \in \mathbb{R}^m$ satisfying $\|\alpha\| < r$ (see footnote [a]) the system of equations (2.4) has a unique solution[b] in U:

$$u^i = f^i(t, x^k, \alpha^k), \qquad |t| < \delta, \tag{2.5}$$

that satisfies the initial conditions

$$\begin{cases} u^i(0) &= f^i(0, x^k, \alpha^k) = x^i, \\ \dfrac{du^i}{dt}(0) &= \left.\dfrac{\partial f^i(t, x^k, \alpha^k)}{\partial t}\right|_{t=0} = \alpha^i. \end{cases} \tag{2.6}$$

Furthermore the functions f^i depend smoothly on the independent variable t and the initial values x^k, α^k.

transformation, a normal coordinate system of a point in M is determined up to a nondegenerate linear transformation.

Fix $\alpha^k = \alpha_0^k$. As t changes, $t\alpha_0^k$ describes a straight line in $T_x(M)$ starting from the origin, and traces a geodesic curve on the manifold starting from x and tangent to the tangent vector (α_0^k). Therefore the equation for this geodesic curve under the normal coordinate system α^i is

$$\alpha^k = t\alpha_0^k, \tag{2.12}$$

where α_0^k is a constant.

Theorem 2.1. *If M is a torsion-free affine connection space, then with respect to a normal coordinate system α^i at the point x, the connection coefficients Γ_{ik}^j are zero at x.*

Proof. Since the geodesic curve $\alpha^i = t\alpha_0^i$ satisfies (2.4) under the normal coordinate system α^i, we have, for any α_0^k,

$$\Gamma_{jk}^i(0)\alpha_0^j\alpha_0^k = 0. \tag{2.13}$$

Since Γ_{jk}^i is symmetric in the lower indices for torsion-free connections, we have

$$\Gamma_{jk}^i(0) = 0, \qquad 1 \le i, j, k \le m. \tag{2.14}$$

□

Theorem 2.2. *For any point x_0 in an affine connection space M, there exists a neighborhood W of x_0 such that every point in W has a normal coordinate neighborhood that contains W.*

Proof. Suppose $(U; u^i)$ is a normal coordinate system at a point x_0. Let

$$U(x_0;\rho) = \left\{ x \in U \mid \sum_{i=1}^m (u^i(x))^2 < \rho^2 \right\}. \tag{2.15}$$

By the above discussion on the solution of equation (2.4), there exists a neighborhood $W = U(x_0; r)$ of x_0 and a positive number δ such that for any $x \in W$ and $\alpha \in \mathbb{R}^m$, $\|\alpha\| < \delta$, there is a unique geodesic curve and if we change the structure of V then the sections ~~with~~ will have the properties ~~of V~~ and structures of V.

now we consider the manifolds with structure of sections (vector bundle) suppose E, q-dimensional real vector ~~bund~~ bundle, and $\Gamma(E)$ is the set of smooth sections of E on M, $\Gamma(E)$ is a real vector space: it is also a $C^\infty(M)$-module.

let

$$S=\begin{pmatrix} S_1 \\ \vdots \\ S_q \end{pmatrix} \qquad w=\begin{pmatrix} w_1^1 & \cdots & w_1^q \\ \vdots & \ddots & \vdots \\ w_q^1 & \cdots & w_q^q \end{pmatrix}$$

$$DS = w \otimes S$$

If $S' = (S_1', \cdots, S_q')^T$

$S' = A \cdot S$ A a transf matrix

$$\begin{aligned} DS' &= dA \otimes S + A \cdot DS \\ &= (dA + A \cdot w) \otimes S \\ &= (dA \cdot A^{-1} + AwA^{-1}) \otimes S' \end{aligned}$$

$$\Rightarrow \quad w' = dA \cdot A^{-1} + AwA^{-1}$$

This is the transformation formula for a connection matrix obviously w is not a tensor field. because it do not satisfy the transformation formular for tensor. It is a most important formula in differential geometry

Date.
now we try to prove a most basic formular of differential Geometry.

Theorem Suppose D is a connection on a vector bundle E and $p \in M$. Then there exist a local frame field S in a coordinate neighborhood of p such that the corresponding connection

. On exterior differentiating the transformation formula of connection, we have

$$dw' \cdot A - w' \wedge dA = dA \wedge w + A \cdot dw$$

we have $\quad dA = w' \cdot A - A \cdot w$

$$\Rightarrow \quad (dw' - w' \wedge w') \cdot A = A \cdot (dw - w \wedge w)$$

. Definition $\Omega = dw - w \wedge w$ is called the curvature matrix of the connection D on U

$$\Omega' = A \cdot \Omega A^{-1}$$

It is worth mentioning that the transformation formula for Ω is the formula for tensor while that for the connection matrix w is not

now we try to induce a very import

(V TM)

now we let s be the natural basis Date.

$\{s_i = \frac{\partial}{\partial u^i},\ 1 \le i \le m\}$ forms a local frame a local frame field of the tangent bundle $T(M)$ on U

$$Ds_i = w_i^j \otimes s_j = \Gamma_{ik}^j du^k \otimes s_j$$

Γ_{ik}^j are smooth functions on U, called

$$\Rightarrow \quad w'^j_i = d\left(\frac{\partial u^p}{\partial w^i}\right)\frac{\partial w^j}{\partial u^p} + \frac{\partial u^p}{\partial w^i}\frac{\partial w^j}{\partial u^q} w_p^q$$

$$\Rightarrow \quad \Gamma'^j_{ik} = \Gamma_{pr}^q \frac{\partial w^j}{\partial u^q}\frac{\partial u^p}{\partial w^i}\frac{\partial u^r}{\partial w^k} + \frac{\partial^2 u^p}{\partial w^i \partial w^k}\cdot\frac{\partial w^j}{\partial u^p}$$

$$X = x^i \frac{\partial}{\partial u^i}$$

$$DX = (dx^i + x^j w_j^i) \otimes \frac{\partial}{\partial u^i}$$

$$= x^i_{,j}\, du^j \otimes \frac{\partial}{\partial u^i}$$

$$x^i_{,j} = \frac{\partial x^i}{\partial u^j} + x^k \Gamma_{kj}^i$$

DX it is a tensor field of type (1,1) on M

DX the absolute differential of X

$$s^{*i} = du^i$$

$$Ds^{*i} = -\Gamma_{jk}^i du^k \otimes du^j$$

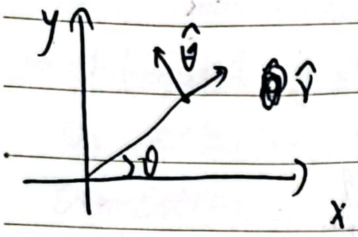
$$D\alpha = (d\alpha_i - \alpha_j w_i^j) \otimes du^i$$

$$= \alpha_{i,j}\, du^j \otimes du^i$$

$$\alpha_{i,j} = \frac{\partial \alpha_i}{\partial u^j} - \alpha_k \Gamma_{ij}^k$$

Date.
now we consider some special case to understand what we have done before

now we consider the polar coordinates.



now we consider the vector $\vec{A}$

$$\vec{A} = A_r \hat{r} + A_\theta \hat{\theta}$$

consider $\frac{\partial \vec{A}}{\partial r}$ or $\frac{\partial \vec{A}}{\partial \theta}$

by defination of the partial differential

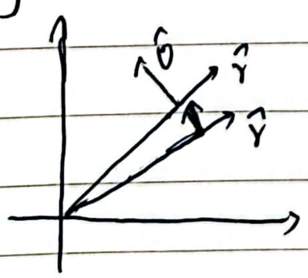
$$\frac{\partial \vec{A}}{\partial r} = \frac{\partial}{\partial r}(A_r \hat{r}) + \frac{\partial}{\partial r}(A_\theta \hat{\theta})$$

$$\overset{?}{=} \frac{\partial}{\partial r} A_r \ \hat{r} + \frac{\partial}{\partial r} A_\theta \ \hat{\theta}$$

$$\frac{\partial \vec{A}}{\partial \theta} = \frac{\partial}{\partial \theta}(A_r \hat{r}) + \frac{\partial}{\partial \theta}(A_\theta \hat{\theta})$$

$$\overset{?}{=} \frac{\partial}{\partial \theta} A_r \ \hat{r} + \frac{\partial}{\partial \theta} A_\theta \ \hat{\theta}$$

in our intuition, the equation we write above may be wrong. consider the following case

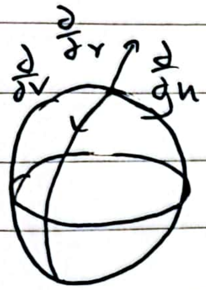


we find that

$$\frac{\partial}{\partial \theta}(\hat{r}) \neq 0 \qquad \frac{\partial}{\partial \theta}(\hat{\theta}) \neq 0$$

$$\frac{\partial}{\partial \theta}(\hat{r}) = \hat{\theta} \qquad \frac{\partial}{\partial \theta}(\hat{\theta}) = -\hat{r}$$

~~it's~~ the most important things we find is that $\frac{\partial}{\partial \theta}(\hat{r})$ $\frac{\partial}{\partial \theta}(\hat{\theta})$ is the linear conbination of the original coordinates. that's the reason we define connection as the linear combination of the ~~section~~ basis of section. If the case we consider above could not persuade us to accept the structure of connection, now we consider another case, consider a sphere imbedded in ~~R~~ E^3



we define a vector field on a sphere imbedded. ~~Gi~~ let the vector field

$$\frac{\partial}{\partial v}\left(\frac{\partial}{\partial v}\right) = - \frac{\partial}{\partial r} \cdots$$

That is the fundamental of the connection
the differential of basis must be the linear combination of basis itself. And the connection connect ~~two~~ two different points vector field. 因为一般来说~~不同~~流行上不同点的 vector field 是无法比较的. 而 connection 则是定义了一种比较方法。

比较物理的理解方法是：我们知道局部平坦性，($\Gamma = 0$) 在这个坐标内（之后会说明）basis 不随 t 变 ⇒ 矢量的加减法是平凡的 so. 任意两点 可以比较（自由 geodesic），但我们要知道在任意 coordinate chart，⇒ 光滑变换，变换后 粒子不再做自由运动（受引力影响）
$\frac{du^i}{dt}$ change $\frac{\partial}{\partial u^i}$ also change 但我们知道广义相对性原理，所有物理规律 在 自由… $\frac{du^i}{dt} = 0$ but $\frac{du'^i}{dt} \neq 0$ 物理上不对
定义 new 导数 $\frac{du'^i}{dt}$ - 引力 $= 0$ ⇒ D …

Date.

$$\frac{d^2 y^i}{dt^2} = 0$$

$$\Downarrow$$

$$\frac{d^2 x^i}{dt^2} + \Gamma \frac{dx}{dt}\frac{dx}{dt} = 0$$

$$\frac{d}{dt}\left(\frac{dy^i}{dt}\right)$$

$$\Rightarrow \frac{d}{dt}\left(\frac{dy^i}{dt}\frac{\partial}{\partial y^i}\right) = 0$$

$$= \frac{d}{dt}\left(\frac{dy^i}{dt}\frac{\partial x^j}{\partial y^i}\frac{\partial}{\partial x^j}\right) = 0$$

$$= \underbrace{\frac{d^2 x^i}{dt^2}\frac{\partial}{\partial x^i}}_{\text{运动}} + \frac{dx^i}{dt}\underbrace{\frac{d}{dt}\left(\frac{\partial}{\partial x^i}\right)}_{\boxed{\Gamma \frac{dx^i}{dt}\frac{\partial}{\partial x^i}}\ \text{引力}} = 0$$

$$D\left(\frac{dx^i}{dt}\frac{\partial}{\partial x^i}\right) = 0$$

物理定律再次相同