Introduction to Tropical Geometry and Its Application in Matroids

Changqian Li

Southern University of Science and Technology

May 27, 2022

Table of Contents

1. Basic Algebraic Geometry
2. Tropical Geometry
3. Matroids

Outline

1. Basic Algebraic Geometry
2. Tropical Geometry
3. Matroids

Basic algebra

Notation:

- $\mathbb{F}$: a field (you may consider it as $\mathbb{R}$ or $\mathbb{C}$).
- $\mathbb{F}[x_1, \ldots, x_n]$: the polynomial ring in n variables over $\mathbb{F}$ (for example, $\mathbb{R}[x]$, $\mathbb{C}[x, y]$).

Basic algebraic geometry

Definition

Let $\mathbb{F}$ be a field. The n-dimensional **affine space** over $\mathbb{F}$ is defined to be the set

$$\mathbb{A}^n_{\mathbb{F}} := \mathbb{F}^n = \{(x_1, \ldots, x_n) : x_i \in \mathbb{F}\}.$$

We may simply denote $\mathbb{A}^n_{\mathbb{F}}$ by $\mathbb{A}^n$.

Example

- The n-dimensional real vector space $\mathbb{A}^n_{\mathbb{R}} = \mathbb{R}^n$.
- The n-dimensional complex vector space $\mathbb{A}^n_{\mathbb{C}} = \mathbb{C}^n$.

Basic algebraic geometry

Definition

Let $\mathbb{F}$ be a field. The n-dimensional **affine space** over $\mathbb{F}$ is defined to be the set

$$\mathbb{A}^n_{\mathbb{F}} := \mathbb{F}^n = \{(x_1, \ldots, x_n) : x_i \in \mathbb{F}\}.$$

We may simply denote $\mathbb{A}^n_{\mathbb{F}}$ by $\mathbb{A}^n$.

Example

- The n-dimensional real vector space $\mathbb{A}^n_{\mathbb{R}} = \mathbb{R}^n$.
- The n-dimensional complex vector space $\mathbb{A}^n_{\mathbb{C}} = \mathbb{C}^n$.

Basic algebraic geometry

Definition

Let $\mathbb{F}$ be a field. Suppose I is an ideal (subset) in $\mathbb{F}[x_1, \ldots, x_n]$. Define

$$V(I) = \{x \in \mathbb{A}^n_{\mathbb{F}} : f(x) = 0 \text{ for all } f \in I\}.$$

A subset $X \subset \mathbb{A}^n_{\mathbb{F}}$ is called an **affine variety** if $X = V(I)$ for some ideal I in $\mathbb{F}[x_1, \ldots, x_n]$.

Example

Example ($n = 1$)

Let $f = a(x - x_1)\dots(x - x_n) \in \mathbb{R}[x]$, where $x_i \in \mathbb{R}$. Then

$$V(f) = \{x_1, \dots, x_n\}$$

is a finite set.

Example ($n = 2$)

Let $f = x - y + 1 \in \mathbb{R}[x, y, z]$. Then

$$V(f) = \{(x, y) \in \mathbb{R}^2 : y = x + 1\}$$

is a line.

Example

Example ($n = 1$)

Let $f = a(x - x_1)\dots(x - x_n) \in \mathbb{R}[x]$, where $x_i \in \mathbb{R}$. Then

$$V(f) = \{x_1, \dots, x_n\}$$

is a finite set.

Example ($n = 2$)

Let $f = x - y + 1 \in \mathbb{R}[x, y, z]$. Then

$$V(f) = \{(x, y) \in \mathbb{R}^2 : y = x + 1\}$$

is a line.

Example

Example ($n = 3$)

Let $f = x, g = y \in \mathbb{R}[x, y]$. Then

$$V(f, g) = V(f) \cap V(g) = \{(0, 0, z) : z \in \mathbb{R}\}$$

is a line.

Note that because

$$V(I) = \bigcap_{f \in I} V(f),$$

a variety $V(I)$ can be considered as the intersection of some hypersurfaces.

Example

Example ($n = 3$)

Let $f = x, g = y \in \mathbb{R}[x, y]$. Then

$$V(f, g) = V(f) \cap V(g) = \{(0, 0, z) : z \in \mathbb{R}\}$$

is a line.

Note that because

$$V(I) = \bigcap_{f \in I} V(f),$$

a variety $V(I)$ can be considered as the intersection of some hypersurfaces.

Outline

1. Basic Algebraic Geometry
2. Tropical Geometry
3. Matroids

Tropical Algebra

Definition

Let $\mathbb{R}$ be the set of real numbers. The **tropical semiring** is defined to be the set $\mathbb{R} \cup \infty$ with **tropical addition** $\oplus$ and **tropical multiplication** $\otimes$ defined as follows:

$$x \oplus y := \min\{x, y\}, \ x \otimes y := x + y.$$

Example

Example

Consider the polynomial

$$F(x) = 2x^3 + 2x^2 + 3x + 5.$$

Its corresponding **tropical polynomial** is

$$\begin{aligned} f(x) &= (2 \otimes x^{\otimes 3}) \oplus (2 \otimes x^{\otimes 2}) \oplus (3 \otimes x) \oplus 5 \\ &= \min\{2 + 3x, 2 + 2x, 3 + x, 5\}. \end{aligned}$$

We may further write $f(x)$ as

$$f(x) = 2 \otimes (x \oplus 0) \otimes (x \oplus 1) \otimes (x \oplus 2).$$

Example

Example

Consider the polynomial

$$F(x) = 2x^3 + 2x^2 + 3x + 5.$$

Its corresponding **tropical polynomial** is

$$\begin{aligned} f(x) &= (2 \otimes x^{\otimes 3}) \oplus (2 \otimes x^{\otimes 2}) \oplus (3 \otimes x) \oplus 5 \\ &= \min\{2 + 3x, 2 + 2x, 3 + x, 5\}. \end{aligned}$$

We may further write $f(x)$ as

$$f(x) = 2 \otimes (x \oplus 0) \otimes (x \oplus 1) \otimes (x \oplus 2).$$

Example

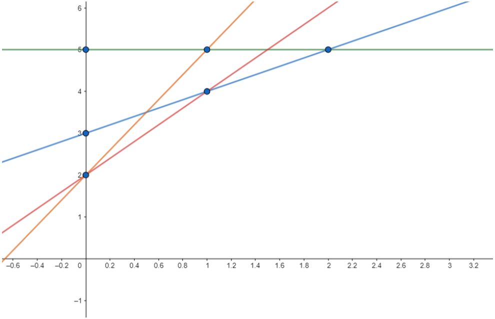

Figure: The graph of $f(x)$.

"Roots" of $f(x)$: the points where the minimum is attained at least twice.

Valuation

Definition

Let $\mathbb{F}$ be a field with a valuation $v : \mathbb{F} \to \mathbb{R} \cup \infty$. We say v is **splitting** if $v : \mathbb{F}^* \to v(\mathbb{F}^*)$ has a right inverse s, where $\mathbb{F}^* = \mathbb{F}\backslash\{0\}$.

Convention: We denote $s(x)$ by t^x.

Residue field

Let $\mathbb{F}$ be a field with a splitting valuation v. Consider the ring of elements with nonnegative valuations

$$R = \{x \in \mathbb{F} : v(x) \geq 0\}.$$

Notice that it is a local ring with a unique maximal ideal

$$m = \{x \in \mathbb{F} : v(x) > 0\}.$$

Denote the residue field R/m by k. Then we have a group homomorphism

$$\mathbb{F}^* \to k^*, \ x \mapsto \overline{xt^{-v(x)}}.$$

We call k the **residue field** of $\mathbb{F}$ induced by v.

Residue field

Let $\mathbb{F}$ be a field with a splitting valuation v. Consider the ring of elements with nonnegative valuations

$$R = \{x \in \mathbb{F} : v(x) \geq 0\}.$$

Notice that it is a local ring with a unique maximal ideal

$$m = \{x \in \mathbb{F} : v(x) > 0\}.$$

Denote the residue field R/m by k. Then we have a group homomorphism

$$\mathbb{F}^* \to k^*, \ x \mapsto \overline{xt^{-v(x)}}.$$

We call k the **residue field** of $\mathbb{F}$ induced by v.

Tropicalization

Definition

Let $\mathbb{F}$ be a field with a valuation v. Suppose $\mathbb{F}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ is the ring of Laurent polynomials over $\mathbb{F}$. Let $f = \sum_{\alpha \in \mathbb{Z}^n} c_\alpha x^\alpha$ be a Laurent polynomial in $\mathbb{F}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$. Then the **tropicalization** of f is defined to be

$$\operatorname{trop} f = \min_{\alpha \in \mathbb{Z}^n} \{ v(c_\alpha) + \alpha \cdot x \}.$$

The **initial form** of f with respect to a point x is defined to be

$$\operatorname{in}_x f = \sum_{v(c_\alpha) + \alpha \cdot x = \operatorname{trop} f(x)} \overline{c_\alpha t^{-v(c_\alpha)}} x^\alpha \in k[x_1^{\pm 1}, \ldots, x_n^{\pm 1}].$$

Tropicalization

Definition

Let $\mathbb{F}$ be a field with a valuation v. Suppose $\mathbb{F}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ is the ring of Laurent polynomials over $\mathbb{F}$. Let $f = \sum_{\alpha \in \mathbb{Z}^n} c_\alpha x^\alpha$ be a Laurent polynomial in $\mathbb{F}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$. Then the **tropicalization** of f is defined to be

$$\operatorname{trop} f = \min_{\alpha \in \mathbb{Z}^n} \{v(c_\alpha) + \alpha \cdot x\}.$$

The **initial form** of f with respect to a point x is defined to be

$$\operatorname{in}_x f = \sum_{v(c_\alpha) + \alpha \cdot x = \operatorname{trop} f(x)} \overline{c_\alpha t^{-v(c_\alpha)}} x^\alpha \in k[x_1^{\pm 1}, \ldots, x_n^{\pm 1}].$$

Initial Ideal

Definition

Let $\mathbb{F}$ be a field with a valuation v, and k be the residue field of $\mathbb{F}$ induced by v. Suppose I is an ideal in $\mathbb{F}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$. The **initial ideal** of I with respect to a point x is defined to be

$$\mathrm{in}_x(I) = \langle \mathrm{in}_x f : f \in I \rangle \subset k[x_1^{\pm 1}, \ldots, x_n^{\pm 1}].$$

Tropical Hypersurface

Definition

Let $\mathbb{F}$ be a field with a valuation v. Suppose $f \in \mathbb{F}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$. Then define

$$V(\operatorname{trop} f) := \left\{ \begin{array}{c} x \in \mathbb{R}^n : \text{the minimum in } \operatorname{trop} f(x) \\ \text{is attained at least twice} \end{array} \right\}.$$

A **tropical hypersurface** is defined to be a set of the form

$$\operatorname{trop} V(f) := V(\operatorname{trop} f)$$

for some $f \in \mathbb{F}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$.

Tropical Hypersurface

Definition

Let $\mathbb{F}$ be a field with a valuation v. Suppose $f \in \mathbb{F}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$. Then define

$$V(\operatorname{trop} f) := \left\{ \begin{array}{c} x \in \mathbb{R}^n : \text{the minimum in } \operatorname{trop} f(x) \\ \text{is attained at least twice} \end{array} \right\}.$$

A **tropical hypersurface** is defined to be a set of the form

$$\operatorname{trop} V(f) := V(\operatorname{trop} f)$$

for some $f \in \mathbb{F}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$.

Example

Example

Let $f = x - y + 1 \in \mathbb{F}[x^{\pm 1}, y^{\pm 1}]$, where $\mathbb{F}$ is a field with a valuation. Then

$$\operatorname{trop} f(x, y) = \min\{x, y, 0\}.$$

So

$$\operatorname{trop} V(f) = \{x = y \leq 0\} \cup \{x = 0 \leq y\} \cup \{y = 0 \leq x\}.$$

Example

Example

Let $f = x - y + 1 \in \mathbb{F}[x^{\pm 1}, y^{\pm 1}]$, where $\mathbb{F}$ is a field with a valuation. Then

$$\operatorname{trop} f(x, y) = \min\{x, y, 0\}.$$

So

$$\operatorname{trop} V(f) = \{x = y \leq 0\} \cup \{x = 0 \leq y\} \cup \{y = 0 \leq x\}.$$

Example

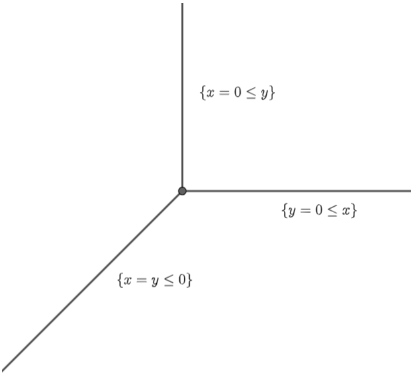

Figure: The graph of trop f.

Tropical Varieties

Definition

Let I be an ideal in $\mathbb{F}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$, and $X = V(I)$ be the corresponding very affine variety in the algebraic torus $\mathbb{T}_{\mathbb{F}}^n = (\mathbb{F}^*)^n$. The **tropicalization** $\operatorname{trop} X$ of X is defined to be

$$\operatorname{trop} X = \bigcap_{f \in I} \operatorname{trop} V(f) \subset \mathbb{R}^n.$$

A **tropical variety** in $\mathbb{R}^n$ is a subset of the form $\operatorname{trop} X$ for some very affine variety X in the algebraic torus $\mathbb{T}^n$.

Tropical Varieties

Definition

Let I be an ideal in $\mathbb{F}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$, and $X = V(I)$ be the corresponding very affine variety in the algebraic torus $\mathbb{T}_{\mathbb{F}}^n = (\mathbb{F}^*)^n$. The **tropicalization** $\operatorname{trop} X$ of X is defined to be

$$\operatorname{trop} X = \bigcap_{f \in I} \operatorname{trop} V(f) \subset \mathbb{R}^n.$$

A **tropical variety** in $\mathbb{R}^n$ is a subset of the form $\operatorname{trop} X$ for some very affine variety X in the algebraic torus $\mathbb{T}^n$.

Fundamental Theorem of Tropical Algebraic Geometry

Theorem (Fundamental Theorem of Tropical Algebraic Geometry)

Let $\mathbb{F}$ be an algebraically closed field with a nontrivial valuation v. Suppose I is an ideal in the ring of Laurent polynomials $\mathbb{F}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$. Then the following subsets of $\mathbb{R}^n$ coincide:

(1) *the tropical variety* $\operatorname{trop} V(I)$;

(2) *the set* $\{x \in \mathbb{R}^n : \operatorname{in}_x(I) \neq \langle 1 \rangle\}$;

(3) *the closure of* $\{(v(x_1), \ldots, v(x_n)) : (x_1, \ldots, x_n) \in V(I)\}$ *in* $\mathbb{R}^n$.

Example

Example

We continue with our previous example of the tropical variety $\operatorname{trop} f$, where $f = x - y + 1$. Then $V(f) = \{(x, x+1) : x \neq 0, 1\}$. If the valuation v is nontrivial, we have

$$
\begin{aligned}
(v(x), v(y)) &= (v(x), v(x+1)) \\
&= \begin{cases} (v(x), 0), & \text{if } v(x) > 0; \\ (v(x), v(x)), & \text{if } v(x) < 0; \\ (0, v(x+1)), & \text{if } v(x) = 0,\ v(x+1) > 0; \\ (0, 0), & \text{otherwise.} \end{cases}
\end{aligned}
$$

Example

Example

We continue with our previous example of the tropical variety trop f, where $f = x - y + 1$. Then $V(f) = \{(x, x+1) : x \neq 0, 1\}$.If the valuation v is nontrivial, we have

$$
\begin{aligned}
(v(x), v(y)) &= (v(x), v(x+1)) \\
&= \begin{cases} (v(x), 0), & \text{if } v(x) > 0; \\ (v(x), v(x)), & \text{if } v(x) < 0; \\ (0, v(x+1)), & \text{if } v(x) = 0,\ v(x+1) > 0; \\ (0, 0), & \text{otherwise.} \end{cases}
\end{aligned}
$$

Example

As x runs over $\mathbb{F}\backslash\{0,1\}$, the closure of such a set coincides with our previous figure of trop f:

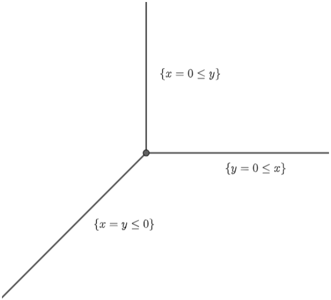

Figure: The graph of trop f.

Summary

Algebraic geometry	**Tropical geometry**
Polynomial f	Tropicalization trop f Initial form $\mathrm{in}_x f$
Ideal I	Initial ideal $\mathrm{in}_x(I)$
Variety V(I)	Tropical variety trop $V(I)$

Table: Algebraic geometry and tropical geometry.

Outline

1. Basic Algebraic Geometry
2. Tropical Geometry
3. Matroids

Polyhedron Geometry

Definition

A **polyhedral cone** C in $\mathbb{R}^n$ is a positive hull of a finite subset:

$$C = \mathrm{pos}(v_1, \ldots, v_r) := \{\sum_{i=1}^{r} \lambda_i v_i \in \mathbb{R}^n : \lambda_i \in \mathbb{R}_{\geq 0} \text{ for all } i\}.$$

A **fan** is a collection Σ of polyhedral cones satisfying the following two conditions:

(1) for any $P \in \Sigma$, each face of P lies in Σ;

(2) for any two elements $A, B \in \Sigma$, if $A \cap B \neq \emptyset$, $A \cap B$ is a face of both.

Polyhedron Geometry

Definition

A **polyhedral cone** C in $\mathbb{R}^n$ is a positive hull of a finite subset:

$$C = \mathrm{pos}(v_1, \ldots, v_r) := \{\sum_{i=1}^{r} \lambda_i v_i \in \mathbb{R}^n : \lambda_i \in \mathbb{R}_{\geq 0} \text{ for all } i\}.$$

A **fan** is a collection Σ of polyhedral cones satisfying the following two conditions:

(1) for any $P \in \Sigma$, each face of P lies in Σ;

(2) for any two elements $A, B \in \Sigma$, if $A \cap B \neq \emptyset$, $A \cap B$ is a face of both.

Hyperplane Arrangements

Let $\mathcal{A} = \{H_i : 0 \leq i \leq n\}$ be an arrangement of $n+1$ hyperplanes with empty intersection in $\mathbb{P}^d$. We are interested in the complement $X = \mathbb{P}^d \setminus \cup \mathcal{A}$, where $\cup \mathcal{A} = \cup_{i=0}^n H_i$.

For each i, write $b_i \in \mathbb{F}^{d+1}$ for a normal vector of the complement H_i. Set $B = \begin{bmatrix} b_0 \dots b_n \end{bmatrix}$. Let $\{a_1, \dots, a_{n-d}\}$ be a basis of $\ker B$, where $a_i = (a_{i0}, \dots, a_{in}) \in \mathbb{F}^{n+1}$.

For each i, let $f_i = \sum_{j=0}^n a_{ij} x_j$. They generate an ideal

$$I = \langle f_1, \dots, f_{n-d} \rangle \subset \mathbb{F}[x_0^{\pm 1}, \dots, x_n^{\pm 1}].$$

Hyperplane Arrangements

Let $\mathcal{A} = \{H_i : 0 \leq i \leq n\}$ be an arrangement of $n+1$ hyperplanes with empty intersection in $\mathbb{P}^d$. We are interested in the complement $X = \mathbb{P}^d \setminus \cup \mathcal{A}$, where $\cup \mathcal{A} = \cup_{i=0}^n H_i$.
For each i, write $b_i \in \mathbb{F}^{d+1}$ for a normal vector of the complement H_i. Set $B = \left[b_0 \ldots b_n\right]$. Let $\{a_1, \ldots, a_{n-d}\}$ be a basis of ker B, where $a_i = (a_{i0}, \ldots, a_{in}) \in \mathbb{F}^{n+1}$.
For each i, let $f_i = \sum_{j=0}^n a_{ij} x_j$. They generate an ideal

$$I = \langle f_1, \ldots, f_{n-d} \rangle \subset \mathbb{F}[x_0^{\pm 1}, \ldots, x_n^{\pm 1}].$$

Hyperplane Arrangements

Let $\mathcal{A} = \{H_i : 0 \leq i \leq n\}$ be an arrangement of $n+1$ hyperplanes with empty intersection in $\mathbb{P}^d$. We are interested in the complement $X = \mathbb{P}^d \setminus \cup \mathcal{A}$, where $\cup \mathcal{A} = \cup_{i=0}^n H_i$.
For each i, write $b_i \in \mathbb{F}^{d+1}$ for a normal vector of the complement H_i. Set $B = \begin{bmatrix} b_0 \dots b_n \end{bmatrix}$. Let $\{a_1, \dots, a_{n-d}\}$ be a basis of $\ker B$, where $a_i = (a_{i0}, \dots, a_{in}) \in \mathbb{F}^{n+1}$.
For each i, let $f_i = \sum_{j=0}^n a_{ij} x_j$. They generate an ideal

$$I = \langle f_1, \dots, f_{n-d} \rangle \subset \mathbb{F}[x_0^{\pm 1}, \dots, x_n^{\pm 1}].$$

Hyperplane Arrangements

Theorem

With notations above. Fixing the torus $\mathbb{T}^n \cong (\mathbb{F}^*)^{n+1}/\mathbb{F}^*$ in $\mathbb{P}^n$, we define a linear map

$$\pi : X = \mathbb{P}^d \setminus \cup \mathcal{A} \to \mathbb{T}^n,\ z \mapsto [b_0 \cdot z : \cdots : b_n \cdot z].$$

This map defines an isomorphism between the arrangement complement X and the subvariety $V(I)$ of $\mathbb{T}^n$.

Here, since I is homogeneous, $V(I)$ is considered as a very affine variety in $\mathbb{T}^n \cong (\mathbb{F}^*)^{n+1}/\mathbb{F}^*$. So trop $V(I)$ is regarded as a subset in $\mathbb{R}^{n+1}/\mathbb{R}\mathbf{1}$, where $\mathbf{1} = (1, \ldots, 1) \in \mathbb{R}^{n+1}$.

Hyperplane Arrangements

Theorem

With notations above. Fixing the torus $\mathbb{T}^n \cong (\mathbb{F}^*)^{n+1}/\mathbb{F}^*$ in $\mathbb{P}^n$, we define a linear map

$$\pi : X = \mathbb{P}^d \backslash \cup \mathcal{A} \to \mathbb{T}^n, \; z \mapsto [b_0 \cdot z : \cdots : b_n \cdot z].$$

This map defines an isomorphism between the arrangement complement X and the subvariety $V(I)$ of $\mathbb{T}^n$.

Here, since I is homogeneous, $V(I)$ is considered as a very affine variety in $\mathbb{T}^n \cong (\mathbb{F}^*)^{n+1}/\mathbb{F}^*$. So trop $V(I)$ is regarded as a subset in $\mathbb{R}^{n+1}/\mathbb{R}\mathbf{1}$, where $\mathbf{1} = (1, \dots, 1) \in \mathbb{R}^{n+1}$.

Fan Structure in $\mathbb{R}^{n+1}/\mathbb{R}\mathbf{1}$

Let $\mathcal{B} = \{b_0, \ldots, b_n\}$. Write $(\mathcal{L}(\mathcal{B}), \subset)$ for the poset consisting of subsets spanned by some of the b_i.

Suppose $\{e_0, \ldots, e_n\}$ is the standard basis of $\mathbb{R}^{n+1}$. Consider a map $\mathcal{P}(\mathcal{L}(\mathcal{B})) \to \mathcal{P}(\mathbb{R}^{n+1})$ defined by

$$\{V_1, \ldots, V_s\} \mapsto \mathrm{pos}(e_{\sigma(V_i)} : 1 \leq i \leq s) + \mathbb{R}\mathbf{1},$$

where $e_{\sigma(V)} = \sum_{i : b_i \in V} e_i$.

Theorem

The image of the above map gives a fan in $\mathbb{R}^{n+1}$. We write $\Delta(\mathcal{B})$ for the image of this fan in $\mathbb{R}^{n+1}/\mathbb{R}\mathbf{1}$.

Fan Structure in $\mathbb{R}^{n+1}/\mathbb{R}\mathbf{1}$

Let $\mathcal{B} = \{b_0, \ldots, b_n\}$. Write $(\mathcal{L}(\mathcal{B}), \subset)$ for the poset consisting of subsets spanned by some of the b_i.

Suppose $\{e_0, \ldots, e_n\}$ is the standard basis of $\mathbb{R}^{n+1}$. Consider a map $\mathcal{P}(\mathcal{L}(\mathcal{B})) \to \mathcal{P}(\mathbb{R}^{n+1})$ defined by

$$\{V_1, \ldots, V_s\} \mapsto \mathrm{pos}(e_{\sigma(V_i)} : 1 \leq i \leq s) + \mathbb{R}\mathbf{1},$$

where $e_{\sigma(V)} = \sum_{i:b_i \in V} e_i$.

Theorem

The image of the above map gives a fan in $\mathbb{R}^{n+1}$. We write $\Delta(\mathcal{B})$ for the image of this fan in $\mathbb{R}^{n+1}/\mathbb{R}\mathbf{1}$.

Fan Structure in $\mathbb{R}^{n+1}/\mathbb{R}\mathbf{1}$

Let $\mathcal{B} = \{b_0, \ldots, b_n\}$. Write $(\mathcal{L}(\mathcal{B}), \subset)$ for the poset consisting of subsets spanned by some of the b_i.

Suppose $\{e_0, \ldots, e_n\}$ is the standard basis of $\mathbb{R}^{n+1}$. Consider a map $\mathcal{P}(\mathcal{L}(\mathcal{B})) \to \mathcal{P}(\mathbb{R}^{n+1})$ defined by

$$\{V_1, \ldots, V_s\} \mapsto \mathrm{pos}(e_{\sigma(V_i)} : 1 \le i \le s) + \mathbb{R}\mathbf{1},$$

where $e_{\sigma(V)} = \sum_{i:b_i \in V} e_i$.

Theorem

The image of the above map gives a fan in $\mathbb{R}^{n+1}$. We write $\Delta(\mathcal{B})$ for the image of this fan in $\mathbb{R}^{n+1}/\mathbb{R}\mathbf{1}$.

Fan structure in $\mathbb{R}^{n+1}/\mathbb{R}\mathbf{1}$

Theorem

Let I be a homogeneous linear ideal in $\mathbb{F}[x_0^{\pm 1}, \dots, x_n^{\pm 1}]$. The tropical variety $\operatorname{trop} V(I)$ *of $V(I)$ equals the support of the fan $\Delta(\mathcal{B})$.*

Matroids

Definition

A **Matroid** is a pair $M = (E, \mathcal{C})$, where E is a finite set and $\mathcal{C}$ is a collection of nonempty subsets of E, called **circuits** of M, such that

(1) no proper subset of a circuit is a circuit;

(2) if C_1, C_2 are distinct circuits and $e \in C_1 \cap C_2$, then $(C_1 \cup C_2) \backslash \{e\}$ contains a circuit.

We may identify E with $\{0, \ldots, n\}$, where $|E| = n + 1$.

Tropical Linear Space

Definition

Let $M = (E, \mathcal{C})$ be a matroid. The **tropical linear space** $\operatorname{trop} M$ of M is the set of vectors $x = (x_0, \ldots, x_n) \in \mathbb{R}^{n+1}$ such that, for any circuit $C \in \mathcal{C}$, the minimum of the numbers x_i is attained at least twice as i ranges over C.

Notice that $\operatorname{trop} M$ is invariant under tropical scalar multiplication: If $x \in \operatorname{trop} M$, $x + \lambda\mathbf{1} \in \operatorname{trop} M$ for all $\lambda \in \mathbb{R}$. So we may consider $\operatorname{trop} M$ as a subset in $\mathbb{R}^{n+1}/\mathbb{R}\mathbf{1}$.

Tropical Linear Space

Definition

Let $M = (E, \mathcal{C})$ be a matroid. The **tropical linear space** $\operatorname{trop} M$ of M is the set of vectors $x = (x_0, \ldots, x_n) \in \mathbb{R}^{n+1}$ such that, for any circuit $C \in \mathcal{C}$, the minimum of the numbers x_i is attained at least twice as i ranges over C.

Notice that $\operatorname{trop} M$ is invariant under tropical scalar multiplication: If $x \in \operatorname{trop} M$, $x + \lambda\mathbf{1} \in \operatorname{trop} M$ for all $\lambda \in \mathbb{R}$. So we may consider $\operatorname{trop} M$ as a subset in $\mathbb{R}^{n+1}/\mathbb{R}\mathbf{1}$.

Fan structure on trop M

We represent each flat F of M by its incidence vector $e_F = \sum_{i \in F} e_i$. For each increasing chain of flats

$$\emptyset \subsetneq F_1 \subsetneq \cdots \subsetneq F_n \subsetneq E,$$

we consider the polyhedral cone in $\mathbb{R}^{n+1}/\mathbb{R}\mathbf{1}$ spanned by their incidence vectors

$$\operatorname{pos}(e_{F_1}, \ldots, e_{F_n}) + \mathbb{R}\mathbf{1} = \{\lambda \mathbf{1} + \sum_{i=1}^{n} \lambda_i e_{F_i} : \lambda_i \geq 0\}.$$

Fan structure on trop M

Theorem

Let M be a matroid on $E = \{0, \ldots, n\}$. The collection of cones

$$\operatorname{pos}(e_{F_1}, \ldots, e_{F_n}) + \mathbb{R}\mathbf{1}$$

where $\emptyset \subsetneq F_1 \subsetneq \cdots \subsetneq F_r \subsetneq E$ runs over all chains of flats in M, form a fan of pure dimension $\rho(M) - 1$ in $\mathbb{R}^{n+1}/\mathbb{R}\mathbf{1}$. Furthermore, the support of this fan equals the tropical linear space trop M.

Summary

Hyperplane Arrangements:

- The arrangement complement $\mathbb{P}^n \setminus \cup \mathcal{A}$ is a subvariety $V(I) \subset \mathbb{T}^n$.
- The tropical variety trop $V(I) \subset \mathbb{R}^{n+1}/\mathbb{R}\mathbf{1}$ is a fan $\Delta(\mathcal{B})$.

Matroids:

- The tropical linear space trop $M \subset \mathbb{R}^{n+1}/\mathbb{R}\mathbf{1}$ is a fan.

Summary

Hyperplane Arrangements:

- The arrangement complement $\mathbb{P}^n \backslash \cup \mathcal{A}$ is a subvariety $V(I) \subset \mathbb{T}^n$.
- The tropical variety trop $V(I) \subset \mathbb{R}^{n+1}/\mathbb{R}\mathbf{1}$ is a fan $\Delta(\mathcal{B})$.

Matroids:

- The tropical linear space trop $M \subset \mathbb{R}^{n+1}/\mathbb{R}\mathbf{1}$ is a fan.

Further Topics

Other topics in tropical geometry:

- The classification of surfaces.
- Moduli spaces.
- Toric geometry.
- ...

Thank You!